Determination of Two-Body Orbital Elements from the Two-Body State Vector

Marc A. Murison

Astronomical Applications Department
U.S. Naval Observatory
Washington, DC

murison@aa.usno.navy.mil
http://aa.usno.navy.mil/murison/

October 23, 1998

1. Initialization

restart

$$\mathrm{alias}\left(r = \mathrm{r}(t),\ r_e = r_e(t),\ r_h = r_h(t),\ v_r = \frac{\partial}{\partial t}\mathrm{r}(t),\ v_{r_e} = \frac{\partial}{\partial t} r_e(t),\ v_{r_h} = \frac{\partial}{\partial t} r_h(t),\ \theta = \theta(t),\ x = \mathrm{x}(t),\ y = \mathrm{y}(t),\ z = \mathrm{z}(t),\ v_x = \frac{\partial}{\partial t}\mathrm{x}(t),\ v_y = \frac{\partial}{\partial t}\mathrm{y}(t),\ v_z = \frac{\partial}{\partial t}\mathrm{z}(t)\right)$$

a utility procedure to convert a vector equation into component equations

```
mat2eqs := proc(Meqn::equation)
local k, eqlist, Mleft, Mright;
    Mleft := evalm(lhs(Meqn));
    Mright := evalm(rhs(Meqn));
    eqlist := [ ];
    for k to rows(Mleft) do eqlist := [op(eqlist), Mleft[k, 1] = Mright[k, 1]] od;
    eval(eqlist)
end
```

2. Two-Body Motion Definitions and Derivations

2.1. angular momentum

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \mathrm{mat}(\mathrm{cross}([x, y, z], [v_x, v_y, v_z]))$$

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} y\, v_z - z\, v_y \\ z\, v_x - x\, v_z \\ x\, v_y - y\, v_x \end{bmatrix}$$

$hvec := \%$

$$h = \sqrt{\mathrm{dot}(\mathrm{rhs}(hvec), \mathrm{rhs}(hvec))}$$

$$h = \sqrt{(y\, v_z - z\, v_y)^2 + (z\, v_x - x\, v_z)^2 + (x\, v_y - y\, v_x)^2}$$

For elliptical motion, we have

$$h = \sqrt{\lambda\, a\, (1 - e^2)}$$

$hesub := \%$

where $\lambda = G\,(m_1 + m_2)$. For hyperbolic motion,

$$h = \sqrt{\lambda\, a\, (e^2 - 1)}$$

$hhsub := \%$

2.2. radius

Elliptical:

$$r = \frac{a\,(1 - e^2)}{1 + e \cos(\theta)}$$

$resub := \%$

Hyperbolic:

$$r = \frac{a\,(e^2 - 1)}{1 + e \cos(\theta)}$$

$rhsub := \%$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos(\Omega) \cos(\omega + \theta) - \sin(\Omega) \sin(\omega + \theta) \cos(i) \\ \sin(\Omega) \cos(\omega + \theta) + \cos(\Omega) \sin(\omega + \theta) \cos(i) \\ \sin(\omega + \theta) \sin(i) \end{bmatrix}$$

$rvec := \%$

$\mathrm{mat2eqs}(rvec)$

$$[x = r\,(\cos(\Omega) \cos(\omega + \theta) - \sin(\Omega) \sin(\omega + \theta) \cos(i)),$$

$$y = r\,(\sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i)),\ z = r\sin(\omega+\theta)\sin(i)]$$

$reqs := \%$

2.3. velocity

radial velocity — elliptical motion

$\frac{\partial}{\partial t}\,resub$

$$v_r = \frac{a\,(1-e^2)\,e\sin(\theta)\left(\frac{\partial}{\partial t}\theta\right)}{(1+e\cos(\theta))^2}$$

$\mathrm{subs}\left(\frac{\partial}{\partial t}\theta = \frac{h}{r^2}, \%\right)$

$$v_r = \frac{a\,(1-e^2)\,e\sin(\theta)\,h}{(1+e\cos(\theta))^2\,r^2}$$

$\mathrm{lhs}(\%) = \mathrm{subs}(resub, \mathrm{rhs}(\%))$

$$v_r = \frac{e\sin(\theta)\,h}{a\,(1-e^2)}$$

$\mathrm{subs}(hesub, \%)$

$$v_r = \frac{e\sin(\theta)\sqrt{\lambda\,a\,(1-e^2)}}{a\,(1-e^2)}$$

$vresub := \%$

radial velocity — hyperbolic motion

$\frac{\partial}{\partial t}\,rhsub$

$$v_r = \frac{a\,(-1+e^2)\,e\sin(\theta)\left(\frac{\partial}{\partial t}\theta\right)}{(1+e\cos(\theta))^2}$$

$\mathrm{subs}\left(\frac{\partial}{\partial t}\theta = \frac{h}{r^2}, \%\right)$

$$v_r = \frac{a\,(-1+e^2)\,e\sin(\theta)\,h}{(1+e\cos(\theta))^2\,r^2}$$

$\text{lhs}(\%) = \text{subs}(\mathit{rhsub}, \text{rhs}(\%))$

$$v_r = \frac{e\sin(\theta)\,h}{a\,(-1+e^2)}$$

$\text{subs}(\mathit{hhsub}, \%)$

$$v_r = \frac{e\sin(\theta)\sqrt{\lambda\,a\,(-1+e^2)}}{a\,(-1+e^2)}$$

$\mathit{vrhsub} := \%$

velocity vector

$\text{map}(\mathit{diff}, \text{lhs}(\mathit{rvec}), t) = \text{map}(\mathit{diff}, \text{evalm}(\text{rhs}(\mathit{rvec})), t)$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_r\,(\cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i)) + r\left(-\cos(\Omega)\sin(\omega+\theta)\left(\frac{\partial}{\partial t}\theta\right) - \sin(\Omega)\cos(\omega+\theta)\left(\frac{\partial}{\partial t}\theta\right)\cos(i)\right) \\ v_r\,(\sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i)) + r\left(-\sin(\Omega)\sin(\omega+\theta)\left(\frac{\partial}{\partial t}\theta\right) + \cos(\Omega)\cos(\omega+\theta)\left(\frac{\partial}{\partial t}\theta\right)\cos(i)\right) \\ v_r\sin(\omega+\theta)\sin(i) + r\cos(\omega+\theta)\left(\frac{\partial}{\partial t}\theta\right)\sin(i) \end{bmatrix}$$

$\text{Collect}\left(\%, \left[\sin(i), r, v_r, \frac{\partial}{\partial t}\theta\right]\right)$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} (-\cos(\Omega)\sin(\omega+\theta) - \sin(\Omega)\cos(\omega+\theta)\cos(i))\left(\frac{\partial}{\partial t}\theta\right)r + v_r\,(\cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i)) \\ (-\sin(\Omega)\sin(\omega+\theta) + \cos(\Omega)\cos(\omega+\theta)\cos(i))\left(\frac{\partial}{\partial t}\theta\right)r \end{bmatrix}$$

$$+ v_r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\Bigg]$$

$$\left[\left(v_r \sin(\omega+\theta) + r\cos(\omega+\theta)\left(\frac{\partial}{\partial t}\theta\right)\right)\sin(i)\right]$$

$$\mathrm{subs}\left(\frac{\partial}{\partial t}\theta = \frac{h}{r^2}, \%\right)$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} =$$

$$\left[\frac{(-\cos(\Omega)\sin(\omega+\theta)-\sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r}\right.$$

$$\left.+ v_r\,(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))\right]$$

$$\left[\frac{(-\sin(\Omega)\sin(\omega+\theta)+\cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r}\right.$$

$$\left.+ v_r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\right]$$

$$\left[\left(v_r \sin(\omega+\theta) + \frac{\cos(\omega+\theta)\,h}{r}\right)\sin(i)\right]$$

$vvec := \%$

2.4. angular momentum revisited

The angular momentum is $h = r\ x\ v$. Hence,

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \mathrm{mat}(\mathrm{cross}(\mathrm{convert}(\mathrm{evalm}(\mathrm{rhs}(rvec)), vector), \mathrm{convert}(\mathrm{rhs}(vvec), vector)))$$

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} =$$

$$\left[r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\left(v_r\sin(\omega+\theta)+\frac{\cos(\omega+\theta)\,h}{r}\right)\right.$$

$$\sin(i) - r\sin(\omega+\theta)\sin(i)\left(\frac{(-\sin(\Omega)\sin(\omega+\theta)+\cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r}\right.$$

$$\left.\left.+ v_r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\right)\right]$$

$$\left[r\sin(\omega+\theta)\sin(i)\left(\frac{(-\cos(\Omega)\sin(\omega+\theta)-\sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))\right) - r\,(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))\left(v_r\sin(\omega+\theta)+\frac{\cos(\omega+\theta)\,h}{r}\right)\sin(i) \right]$$

$$\left[r\,(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))\left(\frac{(-\sin(\Omega)\sin(\omega+\theta)+\cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\right) - r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\left(\frac{(-\cos(\Omega)\sin(\omega+\theta)-\sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))\right)\right]$$

lhs(%) = factormat(map(*collect*, rhs(%), [r, v_r, h, cos(i)], *simplify*))

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = h\begin{bmatrix} \sin(\Omega)\sin(i) \\ -\sin(i)\cos(\Omega) \\ \cos(i) \end{bmatrix}$$

hvec_elmnts := %

2.5. vis viva integral

mat2eqs(*vvec*)

$$\left[v_x = \frac{(-\cos(\Omega)\sin(\omega+\theta)-\sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i)),\ v_y = \frac{(-\sin(\Omega)\sin(\omega+\theta)+\cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i)),\ v_z = \left(v_r\sin(\omega+\theta)+\frac{\cos(\omega+\theta)\,h}{r}\right)\sin(i) \right]$$

$veqs := \%$

$v^2 = \mathrm{add}\left(\mathrm{rhs}(veqs_k)^2, k = 1..3\right)$

$$v^2 = \left(\frac{(-\cos(\Omega)\sin(\omega+\theta) - \sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i))\right)^2 + \left(\frac{(-\sin(\Omega)\sin(\omega+\theta) + \cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i))\right)^2 + \left(v_r\sin(\omega+\theta) + \frac{\cos(\omega+\theta)\,h}{r}\right)^2 \sin(i)^2$$

$\mathrm{collect}(\%, [v_r], simplify)$

$$v^2 = v_r^{\,2} + \frac{h^2}{r^2}$$

$v2eq := \%$

elliptical motion

$\mathrm{simplify}(\mathrm{subs}(vresub, resub, hesub, v2eq))$

$$v^2 = -\frac{\lambda\,(e^2 + 1 + 2\,e\cos(\theta))}{a\,(-1+e^2)}$$

$\mathrm{subs}(\mathrm{isolate}(resub, \cos(\theta)), \%)$

$$v^2 = -\frac{\lambda\left(e^2 - 1 + 2\,\frac{a\,(1-e^2)}{r}\right)}{a\,(-1+e^2)}$$

$\mathrm{collect}(\%, [\lambda, r], simplify)$

$$v^2 = \left(-\frac{1}{a} + 2\,\frac{1}{r}\right)\lambda$$

$v2esub := \%$

hyperbolic motion

$\mathrm{simplify}(\mathrm{subs}(vrhsub, rhsub, hhsub, v2eq))$

$$v^2 = \frac{\lambda\,(e^2 + 1 + 2\,e\cos(\theta))}{a\,(-1 + e^2)}$$

subs(isolate(*rhsub*, cos(θ)), %)

$$v^2 = \frac{\lambda\left(e^2 - 1 + 2\,\frac{a\,(-1 + e^2)}{r}\right)}{a\,(-1 + e^2)}$$

collect(%, [λ, r], *simplify*)

$$v^2 = \left(\frac{1}{a} + 2\,\frac{1}{r}\right)\lambda$$

v2hsub := %

2.6. Laplace vector

The Laplace-Runge-Lenz vector (a constant of two-body motion) is $A = \mathrm{cross}(v, h) - \frac{\lambda\,r}{\mathrm{mag}(r)}$. Recall

rvec

hvec_elmnts

vvec

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i) \\ \sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i) \\ \sin(\omega+\theta)\sin(i) \end{bmatrix}$$

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = h \begin{bmatrix} \sin(\Omega)\sin(i) \\ -\sin(i)\cos(\Omega) \\ \cos(i) \end{bmatrix}$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} =$$

$$\left[\frac{(-\cos(\Omega)\sin(\omega+\theta) - \sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i))\right]$$

$$\left[\frac{(-\sin(\Omega)\sin(\omega+\theta) + \cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r}\right.$$

$$+ v_r \left(\sin(\Omega) \cos(\omega+\theta) + \cos(\Omega) \sin(\omega+\theta) \cos(i)\right)\Bigg]$$

$$\left[\left(v_r \sin(\omega+\theta) + \frac{\cos(\omega+\theta)\, h}{r}\right) \sin(i)\right]$$

Hence,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \mathrm{cross}(\mathrm{convert}(\mathrm{evalm}(\mathrm{rhs}(vvec)), vector), \mathrm{convert}(\mathrm{evalm}(\mathrm{rhs}(hvec\_elmnts)), vector))$$

$$- \lambda\, \mathrm{convert}\left(\frac{\mathrm{rhs}(rvec)}{r}, vector\right)$$

$$\mathrm{Collect}(\mathrm{evalm}(\%), [\lambda, h, v_r], simplify)$$

$$\mathrm{lhs}(\%) = \mathrm{mat}(\mathrm{rhs}(\%))$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} =$$

$$\Bigg[\left(-\cos(\Omega) \cos(\omega+\theta) + \sin(\Omega) \sin(\omega+\theta) \cos(i)\right) \lambda$$

$$+ \frac{\left(\cos(\Omega) \cos(\omega+\theta) - \sin(\Omega) \sin(\omega+\theta) \cos(i)\right) h^2}{r}$$

$$+ \left(\sin(\Omega) \cos(\omega+\theta) \cos(i) + \cos(\Omega) \sin(\omega+\theta)\right) v_r\, h\Bigg]$$

$$\Bigg[\left(-\sin(\Omega) \cos(\omega+\theta) - \cos(\Omega) \sin(\omega+\theta) \cos(i)\right) \lambda$$

$$+ \frac{\left(\sin(\Omega) \cos(\omega+\theta) + \cos(\Omega) \sin(\omega+\theta) \cos(i)\right) h^2}{r}$$

$$+ \left(\sin(\Omega) \sin(\omega+\theta) - \cos(\Omega) \cos(\omega+\theta) \cos(i)\right) v_r\, h\Bigg]$$

$$\left[-\lambda \sin(\omega+\theta) \sin(i) + \frac{\sin(\omega+\theta) \sin(i)\, h^2}{r} - \sin(i) \cos(\omega+\theta)\, v_r\, h\right]$$

$$\mathrm{Collect}(\mathrm{subs}(vresub, resub, hesub, \%), \lambda)$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} =$$

$$[(-\cos(\Omega)\cos(\omega+\theta)+\sin(\Omega)\sin(\omega+\theta)\cos(i)$$
$$+(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))(1+e\cos(\theta))$$
$$+(\sin(\Omega)\cos(\omega+\theta)\cos(i)+\cos(\Omega)\sin(\omega+\theta))\,e\sin(\theta))\,\lambda]$$
$$[(-\sin(\Omega)\cos(\omega+\theta)-\cos(\Omega)\sin(\omega+\theta)\cos(i)$$
$$+(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))(1+e\cos(\theta))$$
$$+(\sin(\Omega)\sin(\omega+\theta)-\cos(\Omega)\cos(\omega+\theta)\cos(i))\,e\sin(\theta))\,\lambda]$$
$$[(-\sin(\omega+\theta)\sin(i)+\sin(\omega+\theta)\sin(i)(1+e\cos(\theta))-\sin(i)\cos(\omega+\theta)\,e\sin(\theta))\,\lambda]$$

factormat(Collect(%, [λ, e, sin(θ), cos(θ)], *factor*))

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \lambda\, e$$

$$[(\cos(\Omega)\cos(\omega+\theta)-\sin(\Omega)\sin(\omega+\theta)\cos(i))\cos(\theta)$$
$$+(\sin(\Omega)\cos(\omega+\theta)\cos(i)+\cos(\Omega)\sin(\omega+\theta))\sin(\theta)]$$
$$[(\sin(\Omega)\cos(\omega+\theta)+\cos(\Omega)\sin(\omega+\theta)\cos(i))\cos(\theta)$$
$$+(\sin(\Omega)\sin(\omega+\theta)-\cos(\Omega)\cos(\omega+\theta)\cos(i))\sin(\theta)]$$
$$[\sin(\omega+\theta)\sin(i)\cos(\theta)-\sin(i)\cos(\omega+\theta)\sin(\theta)]$$

Avec_elmnts := %

The magnitude of the Laplace vector:

convert(evalm(rhs(*Avec_elmnts*)), *vector*)

$A = \text{simplify}(\sqrt{\text{simplify}(\text{dot}(\%, \%))}, assume = positive)$

$$A = \lambda\, e$$

3. Solve for the Orbital Elements

3.1. summary of useful equations

```
hvec; hvec_elmnts; rvec; vvec; hesub; hhsub;
resub; rhsub; vresub; vrhsub; v2eq; v2esub; v2hsub; Avec_elmnts;
```

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} y\,v_z - z\,v_y \\ z\,v_x - x\,v_z \\ x\,v_y - y\,v_x \end{bmatrix}$$

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = h \begin{bmatrix} \sin(\Omega)\sin(i) \\ -\sin(i)\cos(\Omega) \\ \cos(i) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i) \\ \sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i) \\ \sin(\omega+\theta)\sin(i) \end{bmatrix}$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} =$$

$$\left[\frac{(-\cos(\Omega)\sin(\omega+\theta) - \sin(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i))\right]$$

$$\left[\frac{(-\sin(\Omega)\sin(\omega+\theta) + \cos(\Omega)\cos(\omega+\theta)\cos(i))\,h}{r} + v_r\,(\sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i))\right]$$

$$\left[\left(v_r\sin(\omega+\theta) + \frac{\cos(\omega+\theta)\,h}{r}\right)\sin(i)\right]$$

$$h = \sqrt{\lambda\,a\,(1-e^2)}$$

$$h = \sqrt{\lambda\,a\,(-1+e^2)}$$

$$r = \frac{a\,(1-e^2)}{1+e\cos(\theta)}$$

$$r = \frac{a\,(-1+e^2)}{1+e\cos(\theta)}$$

$$v_r = \frac{e\sin(\theta)\sqrt{\lambda\,a\,(1-e^2)}}{a\,(1-e^2)}$$

$$v_r = \frac{e\sin(\theta)\sqrt{\lambda\,a\,(-1+e^2)}}{a\,(-1+e^2)}$$

$$v^2 = v_r^{\,2} + \frac{h^2}{r^2}$$

$$v^2 = \left(-\frac{1}{a} + 2\,\frac{1}{r}\right)\lambda$$

$$v^2 = \left(\frac{1}{a} + 2\,\frac{1}{r}\right)\lambda$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \lambda\, e$$

$$[(\cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i))\cos(\theta)$$
$$+ (\sin(\Omega)\cos(\omega+\theta)\cos(i) + \cos(\Omega)\sin(\omega+\theta))\sin(\theta)]$$
$$[(\sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i))\cos(\theta)$$
$$+ (\sin(\Omega)\sin(\omega+\theta) - \cos(\Omega)\cos(\omega+\theta)\cos(i))\sin(\theta)]$$
$$[\sin(\omega+\theta)\sin(i)\cos(\theta) - \sin(i)\cos(\omega+\theta)\sin(\theta)]$$

3.2. eccentricity

The definition of the Laplace vector is $A = \mathrm{cross}(v, h) - \dfrac{\lambda\, r}{\mathrm{mag}(r)}$. Hence, in component form,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \mathrm{mat}(\mathrm{cross}([v_x, v_y, v_z], [h_x, h_y, h_z])) - \frac{\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}}{\sqrt{x^2 + y^2 + z^2}}$$

evalm(%)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} v_y h_z - v_z h_y - \dfrac{\lambda x}{\sqrt{x^2+y^2+z^2}} \\ v_z h_x - v_x h_z - \dfrac{\lambda y}{\sqrt{x^2+y^2+z^2}} \\ v_x h_y - v_y h_x - \dfrac{\lambda z}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}$$

We also have that

hvec

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} y\,v_z - z\,v_y \\ z\,v_x - x\,v_z \\ x\,v_y - y\,v_x \end{bmatrix}$$

Therefore,

subs(mat2eqs(%), %%)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} v_y\,(x\,v_y - y\,v_x) - v_z\,(z\,v_x - x\,v_z) - \dfrac{\lambda\,x}{\sqrt{x^2 + y^2 + z^2}} \\ v_z\,(y\,v_z - z\,v_y) - v_x\,(x\,v_y - y\,v_x) - \dfrac{\lambda\,y}{\sqrt{x^2 + y^2 + z^2}} \\ v_x\,(z\,v_x - x\,v_z) - v_y\,(y\,v_z - z\,v_y) - \dfrac{\lambda\,z}{\sqrt{x^2 + y^2 + z^2}} \end{bmatrix}$$

Avec := %

Since the magnitude of this vector is $A = \lambda\,e$, we may define an "eccentricity vector":

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \frac{\mathrm{rhs}(Avec)}{\lambda}$$

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \frac{\begin{bmatrix} v_y\,(x\,v_y - y\,v_x) - v_z\,(z\,v_x - x\,v_z) - \dfrac{\lambda\,x}{\sqrt{x^2 + y^2 + z^2}} \\ v_z\,(y\,v_z - z\,v_y) - v_x\,(x\,v_y - y\,v_x) - \dfrac{\lambda\,y}{\sqrt{x^2 + y^2 + z^2}} \\ v_x\,(z\,v_x - x\,v_z) - v_y\,(y\,v_z - z\,v_y) - \dfrac{\lambda\,z}{\sqrt{x^2 + y^2 + z^2}} \end{bmatrix}}{\lambda}$$

from which we obtain the magnitude of the eccentricity, $e = \sqrt{{e_x}^2 + {e_y}^2 + {e_z}^2}$.

An attempt at simplification

convert(evalm(rhs(%)), *vector*)

$\%_1^{\,2} + \%_2^{\,2} + \%_3^{\,2}$

cost(%)

$$20\ \mathit{additions} + 3\ \mathit{divisions} + 42\ \mathit{multiplications} + 66\ \mathit{functions}$$

$$\mathrm{Collect}\left(\%\%, \left[\frac{1}{\sqrt{x^2+y^2+z^2}}, \lambda\right], factor, loc\right)$$

$$1+\left(-2x^2 {v_y}^2+4x v_y y v_x+4x v_z z v_x-2x^2 {v_z}^2-2y^2 {v_z}^2+4 v_z z v_y y-2y^2 {v_x}^2-2z^2 {v_x}^2-2z^2 {v_y}^2\right) \Big/ \left(\lambda\sqrt{x^2+y^2+z^2}\right)+\left(\left({v_x}^2+{v_y}^2+{v_z}^2\right)\left(x^2 {v_y}^2-2x v_y y v_x-2x v_z z v_x+x^2 {v_z}^2+y^2 {v_z}^2-2 v_z z v_y y+y^2 {v_x}^2+z^2 {v_x}^2+z^2 {v_y}^2\right)\right) \Big/ \lambda^2$$

$$x^2 {v_y}^2-2x v_y y v_x-2x v_z z v_x+x^2 {v_z}^2+y^2 {v_z}^2-2 v_z z v_y y+y^2 {v_x}^2+z^2 {v_x}^2+z^2 {v_y}^2$$

$$\mathrm{csquare}\left(\%, \left[{v_x}^2+{v_y}^2+{v_z}^2, x^2+y^2+z^2\right], factor\right)$$

$$\left({v_z}^2+{v_y}^2\right)(x^2+y^2+z^2)+\left(-{v_z}^2+{v_x}^2\right)z^2+\left({v_x}^2-{v_y}^2\right)y^2-2 v_z z v_y y-2x v_z z v_x-2x v_y y v_x$$

$$\mathrm{subs}(\%\% = \%, \%\%\%)$$

$$1+\left(-2x^2 {v_y}^2+4x v_y y v_x+4x v_z z v_x-2x^2 {v_z}^2-2y^2 {v_z}^2+4 v_z z v_y y-2y^2 {v_x}^2-2z^2 {v_x}^2-2z^2 {v_y}^2\right) \Big/ \left(\lambda\sqrt{x^2+y^2+z^2}\right)+\left(\left({v_x}^2+{v_y}^2+{v_z}^2\right)\left(\left({v_z}^2+{v_y}^2\right)(x^2+y^2+z^2)+\left(-{v_z}^2+{v_x}^2\right)z^2+\left({v_x}^2-{v_y}^2\right)y^2-2 v_z z v_y y-2x v_z z v_x-2x v_y y v_x\right)\right) \Big/ \lambda^2$$

$$-2x^2 {v_y}^2+4x v_y y v_x+4x v_z z v_x-2x^2 {v_z}^2-2y^2 {v_z}^2+4 v_z z v_y y-2y^2 {v_x}^2-2z^2 {v_x}^2-2z^2 {v_y}^2$$

$$\mathrm{csquare}\left(\%, \left[{v_x}^2+{v_y}^2+{v_z}^2, x^2+y^2+z^2\right], factor\right)$$

$$\left(-2{v_z}^2-2{v_y}^2\right)(x^2+y^2+z^2)+\left(2{v_z}^2-2{v_x}^2\right)z^2+\left(-2{v_x}^2+2{v_y}^2\right)y^2+4 v_z z v_y y+4x v_z z v_x+4x v_y y v_x$$

$$\mathrm{subs}(\%\% = \%, \%\%\%)$$

$$1+\left(\left(-2\,v_z^2-2\,v_y^2\right)(x^2+y^2+z^2)+\left(2\,v_z^2-2\,v_x^2\right)z^2+\left(-2\,v_x^2+2\,v_y^2\right)y^2 + 4\,v_z\,z\,v_y\,y+4\,x\,v_z\,z\,v_x+4\,x\,v_y\,y\,v_x\right)\Big/\left(\lambda\sqrt{x^2+y^2+z^2}\right)+\left(\left(v_x^2+v_y^2+v_z^2\right)\left(\left(v_z^2+v_y^2\right)(x^2+y^2+z^2)+\left(-v_z^2+v_x^2\right)z^2+\left(v_x^2-v_y^2\right)y^2 - 2\,v_z\,z\,v_y\,y-2\,x\,v_z\,z\,v_x-2\,x\,v_y\,y\,v_x\right)\right)\Big/\lambda^2$$

Collect$\left(\%, \left[\lambda, v_x^2+v_y^2+v_z^2, x^2+y^2+z^2\right], loc\right)$

$$1+\left(\left(-2\,v_z^2-2\,v_y^2\right)\sqrt{x^2+y^2+z^2} + \frac{\left(2\,v_z^2-2\,v_x^2\right)z^2+\left(-2\,v_x^2+2\,v_y^2\right)y^2+4\,v_z\,z\,v_y\,y+4\,x\,v_z\,z\,v_x+4\,x\,v_y\,y\,v_x}{\sqrt{x^2+y^2+z^2}}\right)/\lambda+\left(\left(v_x^2+v_y^2+v_z^2\right)\left(\left(v_z^2+v_y^2\right)(x^2+y^2+z^2)+\left(-v_z^2+v_x^2\right)z^2 + \left(v_x^2-v_y^2\right)y^2-2\,v_z\,z\,v_y\,y-2\,x\,v_z\,z\,v_x-2\,x\,v_y\,y\,v_x\right)\right)\Big/\lambda^2$$

cost(%)

$$26\,\mathit{additions}+69\,\mathit{multiplications}+83\,\mathit{functions}+2\,\mathit{divisions}$$

Oh well.

3.3. semimajor axis

elliptical motion:

isolate$\left(v2esub, \frac{1}{a}\right)$

$$\frac{1}{a}=-\frac{v^2}{\lambda}+2\,\frac{1}{r}$$

hyperbolic motion:

isolate$\left(v2hsub, \frac{1}{a}\right)$

$$\frac{1}{a}=\frac{v^2}{\lambda}-2\,\frac{1}{r}$$

3.4. inclination

We can determine the orbital inclination from the angular momentum vector, calculated from

hvec

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} y\,v_z - z\,v_y \\ z\,v_x - x\,v_z \\ x\,v_y - y\,v_x \end{bmatrix}$$

Since we know that

hvec_elmnts

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = h \begin{bmatrix} \sin(\Omega)\sin(i) \\ -\sin(i)\cos(\Omega) \\ \cos(i) \end{bmatrix}$$

and since the inclination is in the range [0,π], we therefore have

isolate(subs(mat2eqs(*hvec*), mat2eqs(*hvec_elmnts*)$_3$), cos(*i*))

$$\cos(i) = -\frac{-x\,v_y + y\,v_x}{h}$$

For elliptical motion,

subs(*hesub*, %)

$$\cos(i) = -\frac{-x\,v_y + y\,v_x}{\sqrt{\lambda\,a\,(1 - e^2)}}$$

For hyperbolic motion,

subs(*hhsub*, %%)

$$\cos(i) = -\frac{-x\,v_y + y\,v_x}{\sqrt{\lambda\,a\,(-1 + e^2)}}$$

3.5. ascending node

mat2eqs(*hvec_elmnts*)

$$[h_x = h\sin(\Omega)\sin(i),\ h_y = -h\sin(i)\cos(\Omega),\ h_z = h\cos(i)]$$

solve({%$_1$, %$_2$}, {sin(Ω), cos(Ω)})

$$\{\cos(\Omega) = -\frac{h_y}{h \sin(i)}, \sin(\Omega) = \frac{h_x}{h \sin(i)}\}$$

subs(mat2eqs(*hvec*), %)

$$\{\cos(\Omega) = -\frac{z\, v_x - x\, v_z}{h \sin(i)}, \sin(\Omega) = \frac{y\, v_z - z\, v_y}{h \sin(i)}\}$$

Elliptical motion:

subs(*hesub*, %)

$$\left\{\sin(\Omega) = \frac{y\, v_z - z\, v_y}{\sqrt{\lambda\, a\, (1 - e^2)}\, \sin(i)}, \cos(\Omega) = -\frac{z\, v_x - x\, v_z}{\sqrt{\lambda\, a\, (1 - e^2)}\, \sin(i)}\right\}$$

Hyperbolic motion:

subs(*hhsub*, %%)

$$\left\{\cos(\Omega) = -\frac{z\, v_x - x\, v_z}{\sqrt{\lambda\, a\, (-1 + e^2)}\, \sin(i)}, \sin(\Omega) = \frac{y\, v_z - z\, v_y}{\sqrt{\lambda\, a\, (-1 + e^2)}\, \sin(i)}\right\}$$

3.6. true anomaly

circular motion

rvec

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos(\Omega) \cos(\omega + \theta) - \sin(\Omega) \sin(\omega + \theta) \cos(i) \\ \sin(\Omega) \cos(\omega + \theta) + \cos(\Omega) \sin(\omega + \theta) \cos(i) \\ \sin(\omega + \theta) \sin(i) \end{bmatrix}$$

subs(ω = 0, *reqs*)

$$[x = r\, (\cos(\Omega) \cos(\theta) - \sin(\Omega) \sin(\theta) \cos(i)),$$
$$y = r\, (\sin(\Omega) \cos(\theta) + \cos(\Omega) \sin(\theta) \cos(i)), z = r \sin(\theta) \sin(i)]$$

isolate(simplify($\%_1$ cos(Ω) + $\%_2$ sin(Ω)), cos(θ))

$$\cos(\theta) = -\frac{-\cos(\Omega)\, x - \sin(\Omega)\, y}{r}$$

($\%\%_2$ cos(Ω) − $\%\%_1$ sin(Ω)) cos(*i*) + $\%\%_3$ sin(*i*)

$$\cos(i)\, (\cos(\Omega)\, y - \sin(\Omega)\, x) + \sin(i)\, z = \cos(i)\, ($$
$$\cos(\Omega)\, r\, (\sin(\Omega) \cos(\theta) + \cos(\Omega) \sin(\theta) \cos(i))$$
$$- \sin(\Omega)\, r\, (\cos(\Omega) \cos(\theta) - \sin(\Omega) \sin(\theta) \cos(i))) + \sin(i)^2\, r \sin(\theta)$$

collect(isolate(%, sin(θ)), [*r*, cos(*i*)], *simplify*)

$$\sin(\theta) = \frac{-(-\cos(\Omega)\,y + \sin(\Omega)\,x)\cos(i) + \sin(i)\,z}{r}$$

elliptical motion

resub

vresub

$$r = \frac{a\,(1-e^2)}{1+e\cos(\theta)}$$

$$v_r = \frac{e\sin(\theta)\sqrt{\lambda\,a\,(1-e^2)}}{a\,(1-e^2)}$$

sin(θ)

isolate(*vresub*, sin(θ))

$$\sin(\theta) = \frac{v_r\,a\,(1-e^2)}{e\sqrt{\lambda\,a\,(1-e^2)}}$$

$$\mathrm{collect}\left(\mathrm{subs}\left(v_r = \mathrm{dot}\left(\left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right], [v_x, v_y, v_z]\right), \%\right), r\right)$$

$$\sin(\theta) = \frac{(x\,v_x + y\,v_y + z\,v_z)\,a\,(1-e^2)}{e\sqrt{\lambda\,a\,(1-e^2)}\;r}$$

$$\mathrm{subs}\left(r = \sqrt{x^2+y^2+z^2}, \%\right)$$

$$\sin(\theta) = \frac{(x\,v_x + y\,v_y + z\,v_z)\,a\,(1-e^2)}{e\sqrt{\lambda\,a\,(1-e^2)}\,\sqrt{x^2+y^2+z^2}}$$

cos(θ)

isolate(*resub*, cos(θ))

$$\cos(\theta) = \frac{\frac{a\,(1-e^2)}{r} - 1}{e}$$

$$\mathrm{subs}\left(r = \sqrt{x^2+y^2+z^2}, \%\right)$$

$$\cos(\theta) = \frac{\frac{a\,(1-e^2)}{\sqrt{x^2+y^2+z^2}} - 1}{e}$$

hyperbolic motion

rhsub

vrhsub

$$r = \frac{a\,(-1+e^2)}{1+e\cos(\theta)}$$

$$v_r = \frac{e\sin(\theta)\sqrt{\lambda\,a\,(-1+e^2)}}{a\,(-1+e^2)}$$

sin(θ)

isolate(*vrhsub*, sin(θ))

$$\sin(\theta) = \frac{v_r\,a\,(-1+e^2)}{e\sqrt{\lambda\,a\,(-1+e^2)}}$$

$$\text{collect}\left(\text{subs}\left(v_r = \text{dot}\left(\left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right], [v_x, v_y, v_z]\right), \%\right), r\right)$$

$$\sin(\theta) = \frac{(x\,v_x + y\,v_y + z\,v_z)\,a\,(-1+e^2)}{e\sqrt{\lambda\,a\,(-1+e^2)}\;r}$$

$$\text{subs}\left(r = \sqrt{x^2+y^2+z^2}, \%\right)$$

$$\sin(\theta) = \frac{(x\,v_x + y\,v_y + z\,v_z)\,a\,(-1+e^2)}{e\sqrt{\lambda\,a\,(-1+e^2)}\sqrt{x^2+y^2+z^2}}$$

cos(θ)

isolate(*rhsub*, cos(θ))

$$\cos(\theta) = \frac{\frac{a\,(-1+e^2)}{r} - 1}{e}$$

$$\text{subs}\left(r = \sqrt{x^2+y^2+z^2}, \%\right)$$

$$\cos(\theta) = \frac{\frac{a\,(-1+e^2)}{\sqrt{x^2+y^2+z^2}} - 1}{e}$$

3.7. argument of pericenter

rvec

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos(\Omega)\cos(\omega+\theta) - \sin(\Omega)\sin(\omega+\theta)\cos(i) \\ \sin(\Omega)\cos(\omega+\theta) + \cos(\Omega)\sin(\omega+\theta)\cos(i) \\ \sin(\omega+\theta)\sin(i) \end{bmatrix}$$

collect($reqs_1 \cos(\Omega) + reqs_2 \sin(\Omega)$, r, *simplify*)

$$\cos(\Omega)\,x + \sin(\Omega)\,y = \cos(\omega+\theta)\,r$$

isolate(%, $\cos(\omega+\theta)$)

$$\cos(\omega+\theta) = -\frac{-\cos(\Omega)\,x - \sin(\Omega)\,y}{r}$$

collect($(reqs_2 \cos(\Omega) - reqs_1 \sin(\Omega))\cos(i) + reqs_3 \sin(i)$, $[r, \cos(i)]$, *simplify*)

$$\cos(i)\,(\cos(\Omega)\,y - \sin(\Omega)\,x) + \sin(i)\,z = \sin(\omega+\theta)\,r$$

isolate(%, $\sin(\omega+\theta)$)

$$\sin(\omega+\theta) = -\frac{-\cos(i)\,(\cos(\Omega)\,y - \sin(\Omega)\,x) - \sin(i)\,z}{r}$$

3.8. Summary

eccentricity

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} \frac{v_y\,(x\,v_y - y\,v_x) - v_z\,(z\,v_x - x\,v_z)}{\lambda} - \frac{x}{\sqrt{x^2+y^2+z^2}} \\ \frac{v_z\,(y\,v_z - z\,v_y) - v_x\,(x\,v_y - y\,v_x)}{\lambda} - \frac{y}{\sqrt{x^2+y^2+z^2}} \\ \frac{v_x\,(z\,v_x - x\,v_z) - v_y\,(y\,v_z - z\,v_y)}{\lambda} - \frac{z}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}$$

$$e = \sqrt{e_x^2 + e_y^2 + e_z^2}$$

semimajor axis

$$\frac{1}{a}=\frac{2}{r}-\frac{v^2}{\lambda} \text{ or } \frac{1}{a}=\frac{v^2}{\lambda}-\frac{2}{r}$$

<u>inclination</u>

$$\cos(i)=\frac{x\,v_y-y\,v_x}{\sqrt{\lambda\,a\,(1-e^2)}} \text{ or } \cos(i)=\frac{x\,v_y-y\,v_x}{\sqrt{\lambda\,a\,(e^2-1)}}$$

<u>ascending node</u>

$$\left\{\sin(\Omega)=\frac{y\,v_z-z\,v_y}{\sqrt{\lambda\,a\,(1-e^2)}\,\sin(i)}, \cos(\Omega)=\frac{x\,v_z-z\,v_x}{\sqrt{\lambda\,a\,(1-e^2)}\,\sin(i)}\right\}$$

or

$$\left\{\sin(\Omega)=\frac{y\,v_z-z\,v_y}{\sqrt{\lambda\,a\,(e^2-1)}\,\sin(i)}, \cos(\Omega)=\frac{x\,v_z-z\,v_x}{\sqrt{\lambda\,a\,(e^2-1)}\,\sin(i)}\right\}$$

<u>true anomaly</u>

$$\{\sin(\theta)=\frac{(\cos(\Omega)\,y-\sin(\Omega)\,x)\cos(i)+\sin(i)\,z}{r}, \cos(\theta)=\frac{\cos(\Omega)\,x+\sin(\Omega)\,y}{r}\}$$

for circular motion, or

$$\left\{\sin(\theta)=\frac{(x\,v_x+y\,v_y+z\,v_z)\sqrt{a\,(1-e^2)}}{e\,r\sqrt{\lambda}}, \cos(\theta)=\frac{a\,(1-e^2)}{e\,r}-\frac{1}{e}\right\}$$

for elliptical motion, or

$$\left\{\sin(\theta)=\frac{(x\,v_x+y\,v_y+z\,v_z)\sqrt{a\,(e^2-1)}}{e\,r\sqrt{\lambda}}, \cos(\theta)=\frac{a\,(e^2-1)}{e\,r}-\frac{1}{e}\right\}$$

for hyperbolic motion

<u>argument of pericenter</u>

$$\{\sin(\omega+\theta)=\frac{\cos(i)\,(\cos(\Omega)\,y-\sin(\Omega)\,x)+\sin(i)\,z}{r}, \cos(\omega+\theta)=\frac{\cos(\Omega)\,x+\sin(\Omega)\,y}{r}\}$$