Rigid Body Dynamics of a Spinning Truncated Cone Immersed in a Pressure Field

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Euler Equations of Motion for a Rigid Body

restart;
alias$(I = I, \psi = \psi(t), \theta = \theta(t), \phi = \phi(t), \Omega_x = \Omega_x(t), \Omega_y = \Omega_y(t), \Omega_z = \Omega_z(t), \Omega_\phi = \Omega_\phi(t), \Omega_\psi = \Omega_\psi(t), \Omega_\theta = \Omega_\theta(t))$

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Warning, new definition for norm
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In the frame of reference attached to the rotating body (the body frame), the equations of motion are

$EulerEqs$:= mat$\left(I_x\left(\frac{\partial}{\partial t}\Omega_x\right)+(I_z-I_y)\,\Omega_y\,\Omega_z-K_x,\ I_y\left(\frac{\partial}{\partial t}\Omega_y\right)+(I_x-I_z)\,\Omega_x\,\Omega_z-K_y,\ I_z\left(\frac{\partial}{\partial t}\Omega_z\right)+(I_y-I_x)\,\Omega_x\,\Omega_y-K_z\right)$

$$EulerEqs := \begin{bmatrix} I_x\left(\frac{\partial}{\partial t}\Omega_x\right)+(I_z-I_y)\,\Omega_y\,\Omega_z-K_x \\ I_y\left(\frac{\partial}{\partial t}\Omega_y\right)+(I_x-I_z)\,\Omega_x\,\Omega_z-K_y \\ I_z\left(\frac{\partial}{\partial t}\Omega_z\right)+(I_y-I_x)\,\Omega_x\,\Omega_y-K_z \end{bmatrix}$$

latex(*EulerEqs*, "d:/dynamics/precession/EulerEqs.tex")

where I_x, I_y, I_z are the principal moments of inertia of the body, $\Omega_x, \Omega_y, \Omega_z$ are the angular velocities of the body about the principal axes, and K_x, K_y, K_z are the components of the torque acting on the body.

Eulerian Angle Transformation

Rotation Matrix

Construct a rotation matrix that transforms coordinates from the fixed frame (X,Y,Z) to the body frame (x,y,z). First, rotate the coordinates ccw around the Z axis.

$$r1 := \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, rotate ccw around the X' axis.

$$r2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & \sin(\psi) \\ 0 & -\sin(\psi) & \cos(\psi) \end{bmatrix}$$

Next, rotate ccw around the Z" axis.

$$r3 := \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now combine the rotations into a single rotation matrix.

$$R := (p, q, r) \to \mathrm{evalm}((\mathrm{subs}(\theta = r, \mathrm{eval}(r3))\ `\&*`\ \mathrm{subs}(\psi = q, \mathrm{eval}(r2)))\ `\&*`\ \mathrm{subs}(\phi = p, \mathrm{eval}(r1)))$$

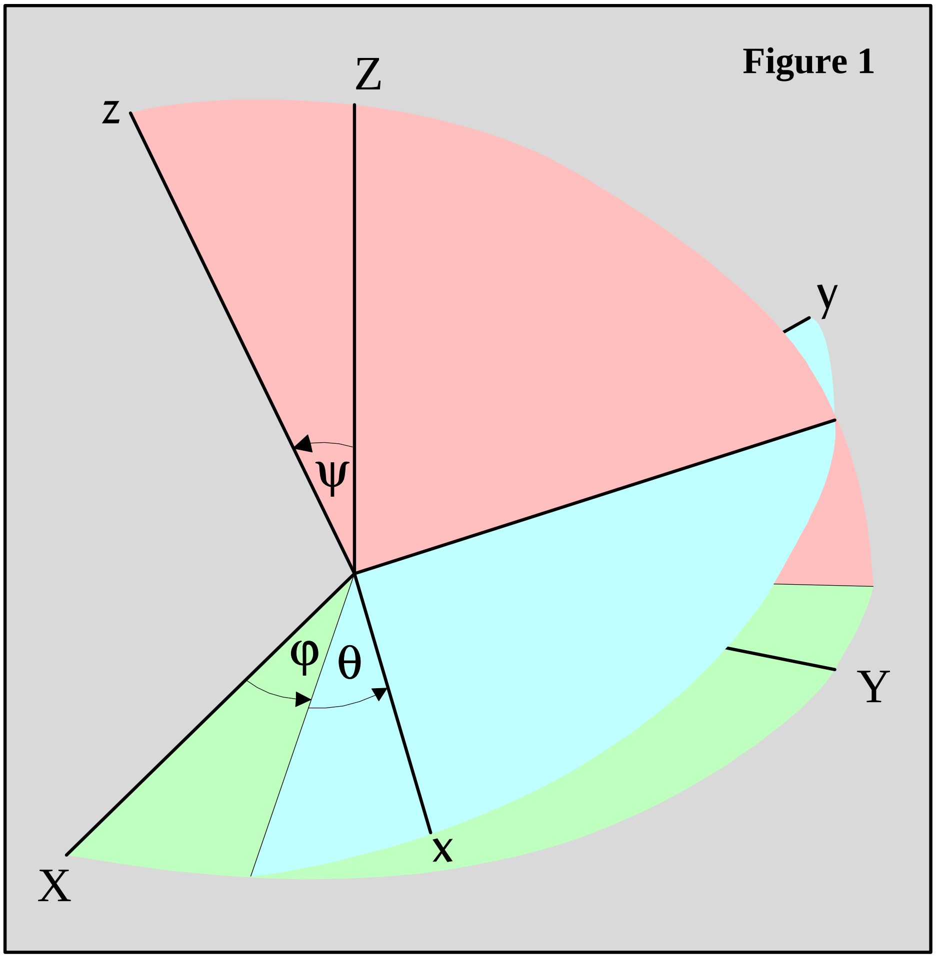
Hence, we have the coordinate transformation

$$\mathrm{mat}(x, y, z) = \mathrm{R}(\phi, \psi, \theta)\ \&*\ \mathrm{mat}(X, Y, Z)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} [\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi), \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi), \sin(\theta)\sin(\psi)] \\ [-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi), -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi), \\ \cos(\theta)\sin(\psi)] \\ [\sin(\psi)\sin(\phi), -\sin(\psi)\cos(\phi), \cos(\psi)] \end{matrix}\ \&*\ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

latex(%, "d:/dynamics/precession/FixedToBody.tex")

The diagram below illustrates the three rotations.



Angular Velocity Vector in the Body Frame

The angular velocity vector may be decomposed into components along each of the rotation axes used to construct the transformation matrix. If we transform those components to the body frame, then we can express the angular velocity vector in the body frame in terms of the Euler angles (ϕ, ψ, θ). The component of Ω along the first rotation axis, as viewed in the body frame, is

$$\omega_{\phi} := \mathrm{evalm}(R(0, \psi, \theta)\ \&*\ \mathrm{mat}(0, 0, 1))\left(\frac{\partial}{\partial t}\phi\right)$$

$$\omega_{\phi} := \begin{bmatrix} \sin(\theta)\sin(\psi) \\ \cos(\theta)\sin(\psi) \\ \cos(\psi) \end{bmatrix}\left(\frac{\partial}{\partial t}\phi\right)$$

The component along the Y' axis is, in the body frame,

$$\omega_{\psi} := \mathrm{evalm}(R(0, 0, \theta)\ \&*\ \mathrm{mat}(1, 0, 0))\left(\frac{\partial}{\partial t}\psi\right)$$

$$\omega_\psi := \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \end{bmatrix} \left(\frac{\partial}{\partial t}\psi\right)$$

Finally, the component along the Z" axis is simply

$$\omega_\theta := \mathrm{mat}(0, 0, 1)\left(\frac{\partial}{\partial t}\theta\right)$$

$$\omega_\theta := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\frac{\partial}{\partial t}\theta\right)$$

Hence, we have the angular velocity vector in the body frame,

$$\Omega = \mathrm{mat}\left(\mathrm{seq}\left(\mathrm{evalm}(\omega_\phi)_{i,\,1} + \mathrm{evalm}(\omega_\psi)_{i,\,1} + \mathrm{evalm}(\omega_\theta)_{i,\,1},\ i = 1\,..\,3\right)\right)$$

$$\Omega = \begin{bmatrix} \left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta) \\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta) \\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right) \end{bmatrix}$$

latex(%, "d:/dynamics/precession/OmegaBodyFrame.tex")

$Obody$:= convert(rhs(%), $vector$)

Rigid Body Equations — General Case

Equations of Motion

Substituting back into the Euler equations, we find

$xyz := [x, y, z]$

$MI := [I_x, I_y, I_z]$

$$\mathrm{subs}\left(\mathrm{seq}\left(\Omega_{xyz_k}(t) = Obody_k,\ k = 1\,..\,3\right), \mathrm{evalm}(EulerEqs)\right)$$

$$\Bigg[I_x\left(\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\right)$$
$$+ (I_z - I_y)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right)\right) - K_x\Bigg]$$

$$\Bigg[I_y\left(\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)\right)$$

$$+\left(I_x - I_z\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right)\right) - K_y\Bigg]$$

$$\Bigg[I_z\left(\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right)\right)\right)$$

$$+\left(I_y - I_x\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)$$

$$- K_z\Bigg]$$

$$\left[\mathrm{seq}\left(\mathrm{collect}\left(\frac{\%_{i,\,1}}{MI_i}, \left[\frac{\partial^2}{\partial t\,\partial t}\theta, \frac{\partial^2}{\partial t\,\partial t}\psi, \frac{\partial^2}{\partial t\,\partial t}\phi, \sin(\psi), \frac{\partial}{\partial t}\psi, \sin(\theta), \cos(\theta), \mathit{diff}\right], \mathit{simplify}\right), i = 1\,..\,3\right)\right]$$

$$\Bigg[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)$$

$$+\left(\frac{\left(I_z - I_y\right)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_x} + \frac{\left(I_x + I_z - I_y\right)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}\right)\cos(\theta)\sin(\psi)$$

$$+\left(\frac{\cos(\psi)\left(I_x - I_z + I_y\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x} - \frac{\left(I_x + I_z - I_y\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}\right)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_x}{I_x}, -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)$$

$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)$$

$$+\left(\frac{\left(I_x - I_z\right)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_y} + \frac{\left(-I_y + I_x - I_z\right)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}\right)\sin(\theta)\sin(\psi)$$

$$+\left(\frac{\cos(\psi)\left(I_x - I_z + I_y\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y} + \frac{\left(-I_y + I_x - I_z\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}\right)\cos(\theta)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_y}{I_y}, \left(\frac{\partial^2}{\partial t^2}\theta\right)$$

$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi) - \frac{\left(-I_y + I_x\right)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\theta)\sin(\theta)\sin(\psi)^2}{I_z}$$

$$+\left(\frac{\left(-I_y + I_x\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)^2}{I_z} - \frac{\left(-I_y + I_x\right)\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)^2}{I_z} - \left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\psi)$$

$$+\frac{(-I_y+I_x)\cos(\theta)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)^2}{I_z}-\frac{K_z}{I_z}\Bigg]$$

$tmp := \%$

Clean up the third equation a bit

$\mathrm{algsubs}(\cos(\theta)\sin(\theta)=A, tmp_3)$

$$\mathrm{algsubs}\left(\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)=\frac{B}{\frac{\partial}{\partial t}\phi}, \%\right)$$

$\mathrm{algsubs}(I_x-I_y=C\,I_z, \%)$

$$\left(\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)^2}{I_z}-\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)^2}{I_z}-\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\psi)$$
$$-\frac{(-I_y+I_x)A\left(-\left(\frac{\partial}{\partial t}\psi\right)^2+\left(\frac{\partial}{\partial t}\phi\right)^2\sin(\psi)^2\right)}{I_z}+\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\frac{K_z}{I_z}$$

$$-A\left(-\left(\frac{\partial}{\partial t}\psi\right)^2+\left(\frac{\partial}{\partial t}\phi\right)^2\sin(\psi)^2\right)C+\frac{B\left(-\cos(\theta)^2 C\,I_z+\sin(\theta)^2 C\,I_z-I_z\right)}{I_z}+\left(\frac{\partial^2}{\partial t^2}\theta\right)$$
$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\frac{K_z}{I_z}$$

$$\mathrm{Collect}\left(\%, \left[\frac{\partial^2}{\partial t\,\partial t}\theta, \frac{\partial^2}{\partial t\,\partial t}\phi, A, B, C, \left(\frac{\partial}{\partial t}\phi\right)^2, \cos(\theta)\right], simplify, alg\right)$$

$$\mathrm{subs}\left(A=\cos(\theta)\sin(\theta), B=\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right), C=\frac{I_x-I_y}{I_z}, \%\right)$$

$$\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)+\frac{\left((-1+\cos(\psi)^2)\left(\frac{\partial}{\partial t}\phi\right)^2+\left(\frac{\partial}{\partial t}\psi\right)^2\right)(-I_y+I_x)\cos(\theta)\sin(\theta)}{I_z}$$
$$+\left(\frac{(-2\cos(\theta)^2+1)(-I_y+I_x)}{I_z}-1\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_z}{I_z}$$

$tmp_3 := \%$

$BodyEqs := \text{eval}(tmp)$

latex(*BodyEqs*, "d:/dynamics/precession/RigidBodyEqs.tex")

mat(*BodyEqs*)

$$\left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{(I_z-I_y)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_x}+\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}\right)\cos(\theta)\sin(\psi)+\left(\frac{\cos(\psi)(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}-\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}\right)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_x}{I_x}\right]$$

$$\left[-\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\frac{(I_x-I_z)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_y}+\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}\right)\sin(\theta)\sin(\psi)+\left(\frac{\cos(\psi)(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}+\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}\right)\cos(\theta)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_y}{I_y}\right]$$

$$\left[\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)+\frac{\left((-1+\cos(\psi)^2)\left(\frac{\partial}{\partial t}\phi\right)^2+\left(\frac{\partial}{\partial t}\psi\right)^2\right)(-I_y+I_x)\cos(\theta)\sin(\theta)}{I_z}+\left(\frac{(-2\cos(\theta)^2+1)(-I_y+I_x)}{I_z}-1\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_z}{I_z}\right]$$

Rigid Body Equations — Symmetric Top

General Motion of a Symmetric Top

Equations of Motion

$SymTop := \mathrm{subs}(I_x = I_y, I_y = I_{xy}, BodyEqs)$

$$SymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{(I_z - I_{xy})\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_{xy}} + \frac{I_z\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}\right)\cos(\theta)\sin(\psi) + \left(\frac{\cos(\psi)(2 I_{xy} - I_z)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}} - \frac{I_z\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}}\right)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_x}{I_{xy}}, -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi) + \left(\frac{(I_{xy} - I_z)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_{xy}} - \frac{I_z\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}\right)\sin(\theta)\sin(\psi) + \left(\frac{\cos(\psi)(2 I_{xy} - I_z)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}} - \frac{I_z\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}}\right)\cos(\theta)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_y}{I_{xy}}, \left(\frac{\partial^2}{\partial t^2}\theta\right) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi) - \left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_z}{I_z}\right]$$

$SymTop := \left[\mathrm{seq}\left(\mathrm{collect}\left(\mathrm{simplify}\left(SymTop_k, \left\{1 - \frac{I_z}{I_{xy}} = \beta\right\}, [I_z]\right), \left[\frac{\partial^2}{\partial t\,\partial t}\psi, \frac{\partial^2}{\partial t\,\partial t}\phi, \sin(\theta), \cos(\theta), \sin(\psi), \frac{\partial}{\partial t}\psi, diff\right], factor\right), k = 1 .. 3\right)\right]$

$$SymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\phi\right) + (-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta) + \left(-\cos(\psi)\,\beta\left(\frac{\partial}{\partial t}\phi\right)^2 + (1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta) - \frac{K_x}{I_{xy}}, -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi) + \left(\cos(\psi)\,\beta\left(\frac{\partial}{\partial t}\phi\right)^2 + (-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta)$$

$$+\left(\cos(\psi)\,(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)-\frac{K_y}{I_{xy}},$$

$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)+\frac{K_z}{I_{xy}\,(-1+\beta)}\Bigg]$$

latex($SymTop_1$, "d:/dynamics/precession/SymTop1.tex");

latex($SymTop_2$, "d:/dynamics/precession/SymTop2.tex");

latex($SymTop_3$, "d:/dynamics/precession/SymTop3.tex")

Conversion to a System of First-Order ODEs

We may write the equations of motion as a system of first-order differential equations.

$$subslist := \left[\frac{\partial}{\partial t}\phi=\Omega_{\phi},\ \frac{\partial}{\partial t}\psi=\Omega_{\psi},\ \frac{\partial}{\partial t}\theta=\Omega_{\theta}\right]$$

subs(*subslist*, *SymTop*)

$$\operatorname{solve}\left(\{\operatorname{op}(\%)\},\ \{\frac{\partial}{\partial t}\Omega_{\phi},\ \frac{\partial}{\partial t}\Omega_{\psi},\ \frac{\partial}{\partial t}\Omega_{\theta}\}\right)$$

collect(%, [I_{xy}, Ω_{ψ}, sin(ψ), Ω_{θ}], *factor*)

$$\Bigg\{\frac{\partial}{\partial t}\Omega_{\phi}=\frac{((1-\beta)\,\Omega_{\theta}-\cos(\psi)\,(1+\beta)\,\Omega_{\phi})\,\Omega_{\psi}}{\sin(\psi)}+\frac{\cos(\theta)\,K_y+K_x\sin(\theta)}{\sin(\psi)\,I_{xy}},\ \frac{\partial}{\partial t}\Omega_{\theta}=$$

$$\left(-\Omega_{\phi}\,\beta\sin(\psi)+\frac{(-1+\beta)\cos(\psi)\,\Omega_{\theta}+(1+\beta)\,\Omega_{\phi}}{\sin(\psi)}\right)\Omega_{\psi}$$

$$+\frac{-\dfrac{K_z}{-1+\beta}-\dfrac{(\cos(\theta)\,K_y+K_x\sin(\theta))\cos(\psi)}{\sin(\psi)}}{I_{xy}},$$

$$\frac{\partial}{\partial t}\Omega_{\psi}=\left(\cos(\psi)\,\beta\,\Omega_{\phi}^{\,2}+(-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi)+\frac{-K_y\sin(\theta)+\cos(\theta)\,K_x}{I_{xy}}\Bigg\}$$

foo := %

$$\operatorname{select}\left(has,\ foo,\ \frac{\partial}{\partial t}\Omega_{\phi}\right)$$

collect(op(%) sin(ψ), [I_{xy}, Ω_{ψ}, Ω_{θ}], *factor*)

$foo := (foo \text{ minus } \%\%) \text{ union } \{\%\}$

$\text{select}\left(has, foo, \frac{\partial}{\partial t} \Omega_\theta\right)$

$\text{collect}(\text{sinfix}(\text{op}(\%) \sin(\psi), \psi), [I_{xy}, \Omega_\psi, \Omega_\phi, K_z], factor)$

$foo := (foo \text{ minus } \%\%) \text{ union } \{\%\}$

$$foo := \left\{ \sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{\cos(\theta)\,K_y + K_x \sin(\theta)}{I_{xy}}, \right.$$
$$\frac{\partial}{\partial t}\Omega_\psi = \left(\cos(\psi)\,\beta\,\Omega_\phi^{\ 2} + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) + \frac{-K_y \sin(\theta) + \cos(\theta)\,K_x}{I_{xy}},$$
$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi$$
$$\left. + \frac{-\frac{\sin(\psi)\,K_z}{-1+\beta} - (\cos(\theta)\,K_y + K_x \sin(\theta))\cos(\psi)}{I_{xy}} \right\}$$

$FirstOrderODEsK := \left[\frac{\partial}{\partial t}\phi = \Omega_\phi, \frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\theta = \Omega_\theta, \text{op}\left(\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\phi\right)\right),\right.$
$\left.\text{op}\left(\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\psi\right)\right), \text{op}\left(\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\theta\right)\right)\right]$

$\text{mat}(FirstOrderODEsK)$

$$\left[\frac{\partial}{\partial t}\phi = \Omega_\phi\right]$$
$$\left[\frac{\partial}{\partial t}\psi = \Omega_\psi\right]$$
$$\left[\frac{\partial}{\partial t}\theta = \Omega_\theta\right]$$
$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{\cos(\theta)\,K_y + K_x \sin(\theta)}{I_{xy}}\right]$$
$$\left[\frac{\partial}{\partial t}\Omega_\psi = \left(\cos(\psi)\,\beta\,\Omega_\phi^{\ 2} + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) + \frac{-K_y \sin(\theta) + \cos(\theta)\,K_x}{I_{xy}}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right)=((\cos(\psi)^2\beta+1)\Omega_\phi+(-1+\beta)\cos(\psi)\Omega_\theta)\Omega_\psi + \frac{-\frac{\sin(\psi)K_z}{-1+\beta}-(\cos(\theta)K_y+K_x\sin(\theta))\cos(\psi)}{I_{xy}}\right]$$

latex(%, "d:/dynamics/precession/FirstOrderODEsK.tex")

Force-Free Motion of a Symmetric Top

$FFSymTop := \mathrm{subs}(K_x=0, K_y=0, K_z=0, SymTop)$

$$FFSymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta) + \left(-\cos(\psi)\beta\left(\frac{\partial}{\partial t}\phi\right)^2+(1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta), -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\cos(\psi)\beta\left(\frac{\partial}{\partial t}\phi\right)^2+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta) + \left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta), \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)\right]$$

We notice that the third equation,

$FFSymTop_3$

$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)$$

can be written as

$$FFSymTop_3 = \frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)$$

$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)=\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)$$

rhs(%) − lhs(%)

$$0$$

Hence, $\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)=const$

This is the projection of the angular velocity onto the symmetry axis. Recall the components of Ω in the body frame:

$\mathrm{mat}(\Omega_x, \Omega_y, \Omega_z) = \mathrm{convert}(Obody, matrix)$

$$\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix}=\begin{bmatrix}\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi)-\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\end{bmatrix}$$

We see that the angular velocity about the symmetry axis, Ω_z, is a constant of the motion. We also have conservation of energy,

$$E=\frac{I_x\,\Omega_x^{\,2}+I_y\,\Omega_y^{\,2}+I_z\,\Omega_z^{\,2}}{2}$$

$KE := \%$

Substituting the components of Ω in terms of the Euler angles, we have

$\mathrm{collect}\left(\mathrm{subs}\left(\mathrm{seq}\left(\Omega_{xyz_k}(t) = Obody_k, k = 1 .. 3\right), KE\right), [diff, \sin(\psi)], factor\right)$

$KE := \%$

$$\begin{aligned}E=&\left(\left(\frac{1}{2}I_y\cos(\theta)^2+\frac{1}{2}I_x\sin(\theta)^2\right)\sin(\psi)^2+\frac{1}{2}I_z\cos(\psi)^2\right)\left(\frac{\partial}{\partial t}\phi\right)^2\\ &+\left(\sin(\theta)\cos(\theta)\,(-I_y+I_x)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+I_z\left(\frac{\partial}{\partial t}\theta\right)\cos(\psi)\right)\left(\frac{\partial}{\partial t}\phi\right)\\ &+\left(\frac{1}{2}I_x\cos(\theta)^2+\frac{1}{2}I_y\sin(\theta)^2\right)\left(\frac{\partial}{\partial t}\psi\right)^2+\frac{1}{2}I_z\left(\frac{\partial}{\partial t}\theta\right)^2\end{aligned}$$

$subslist := \left[\frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\phi = \Omega_\phi, \frac{\partial}{\partial t}\theta = \Omega_\theta\right]$

$\mathrm{subs}(subslist, I_x = I_y, I_y = I_{xy}, KE)$

$$E = \left(\left(\frac{1}{2} I_{xy} \cos(\theta)^2 + \frac{1}{2} I_{xy} \sin(\theta)^2\right) \sin(\psi)^2 + \frac{1}{2} I_z \cos(\psi)^2\right) \Omega_\phi^{\ 2} + I_z \Omega_\theta \cos(\psi) \Omega_\phi + \left(\frac{1}{2} I_{xy} \cos(\theta)^2 + \frac{1}{2} I_{xy} \sin(\theta)^2\right) \Omega_\psi^{\ 2} + \frac{1}{2} I_z \Omega_\theta^{\ 2}$$

$$\operatorname{collect}\left(\operatorname{simplify}\left(\frac{\%}{I_{xy}}\right), [\Omega_\phi, \Omega_\psi, \Omega_\theta, I_{xy}], \mathit{simplify}\right)$$

$$\frac{E}{I_{xy}} = \left(\frac{1}{2} - \frac{1}{2}\cos(\psi)^2 + \frac{1}{2}\frac{I_z \cos(\psi)^2}{I_{xy}}\right) \Omega_\phi^{\ 2} + \frac{I_z \Omega_\theta \cos(\psi) \Omega_\phi}{I_{xy}} + \frac{1}{2}\Omega_\psi^{\ 2} + \frac{1}{2}\frac{I_z \Omega_\theta^{\ 2}}{I_{xy}}$$

$$\operatorname{collect}(\operatorname{algsubs}(I_z = I_{xy}(1-\beta), \%), [\Omega], \mathit{factor})$$

$$\frac{E}{I_{xy}} = \Omega_\phi^{\ 2}\left(\frac{1}{2} - \frac{1}{2}\cos(\psi)^2 \beta\right) - (-1+\beta)\cos(\psi)\,\Omega_\theta \Omega_\phi + \frac{1}{2}\Omega_\psi^{\ 2} + \left(\frac{1}{2} - \frac{1}{2}\beta\right)\Omega_\theta^{\ 2}$$

$KE := \%$

Steady Precession Solution

Suppose we look for a solution such that ψ is constant. Then we have

$$\operatorname{eval}\left(\operatorname{subs}\left(\frac{\partial}{\partial t}\psi = 0, \mathit{FFSymTop}\right)\right)$$

$$\left[\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(-\cos(\psi)\,\beta\left(\frac{\partial}{\partial t}\phi\right)^2 + (1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta),\right.$$
$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi) + \left(\cos(\psi)\,\beta\left(\frac{\partial}{\partial t}\phi\right)^2 + (-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta),$$
$$\left.\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi) + \left(\frac{\partial^2}{\partial t^2}\theta\right)\right]$$

The first two equations can be combined to yield

$$\operatorname{collect}\left(\frac{\%_1 \sin(\theta) + \%_2 \cos(\theta)}{\sin(\psi)}, [\mathit{diff}], \mathit{simplify}\right) = 0$$

$$\frac{\partial^2}{\partial t^2}\phi = 0$$

Hence, solutions with $\psi = \mathit{const}$ force a constant precession, $\frac{\partial}{\partial t}\phi = \mathit{const}$. From the third equation, we then see that $\frac{\partial}{\partial t}\theta = const$ as well.

$\mathrm{subs}\left(\frac{\partial^2}{\partial t\,\partial t}\phi = 0, \%\%_1\right)$

$\mathrm{solve}\left(\%, \frac{\partial}{\partial t}\phi\right)$

$\frac{\partial}{\partial t}\phi = \%_2$

$$\frac{\partial}{\partial t}\phi = -\frac{\left(\frac{\partial}{\partial t}\theta\right)(-1+\beta)}{\cos(\psi)\,\beta}$$

$\mathrm{normal}\left(\mathrm{subs}\left(\beta = \frac{I_{xy} - I_z}{I_{xy}}, \%\right)\right)$

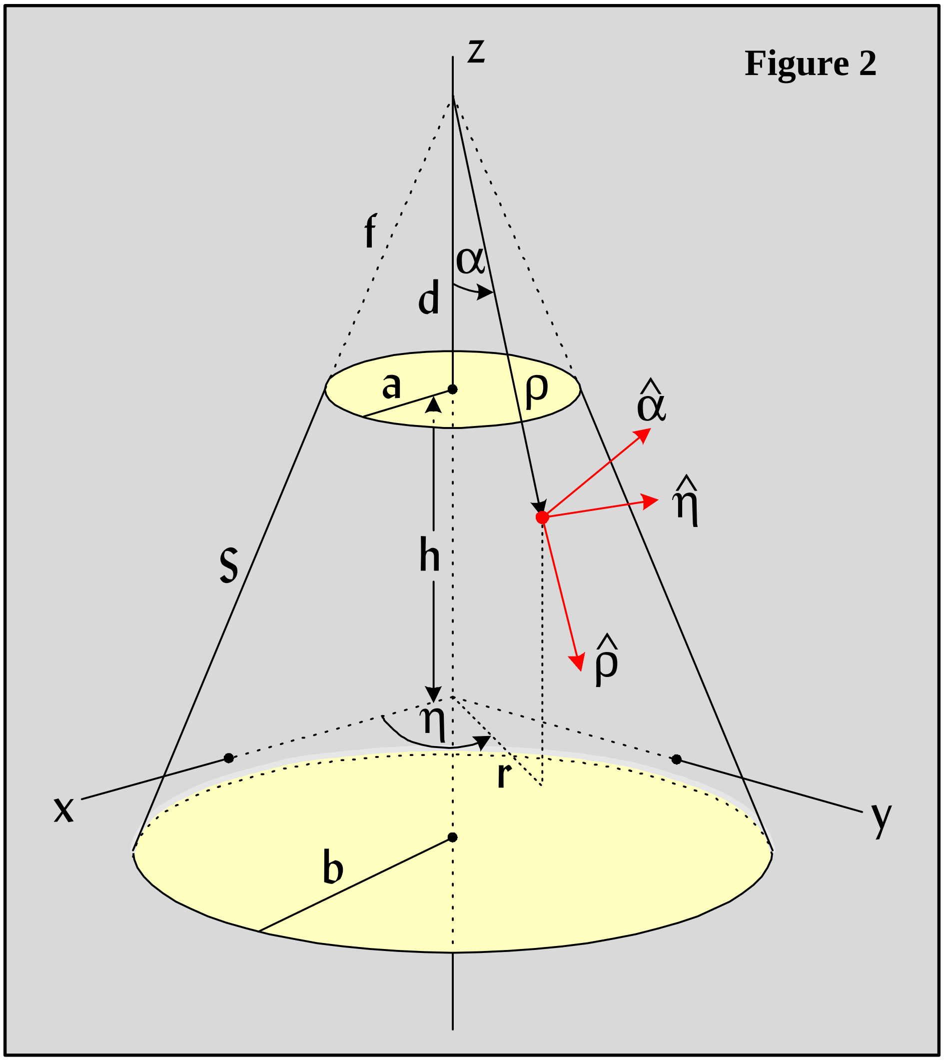
$$\frac{\partial}{\partial t}\phi = -\frac{\left(\frac{\partial}{\partial t}\theta\right) I_z}{\cos(\psi)\,(-I_{xy} + I_z)}$$

A Spinning Truncated Cone Embedded in a Pressure Field

Consider a truncated cone with small and large radii a and b and cone opening angle α. Let the center of mass reside on the (local) z axis, a distance h below the smaller flat surface (the top surface of the truncated cone). Then the equation for the conical surface is $\tan\alpha = \frac{r-a}{h-z}$, or $x^2 + y^2 - (a + (h-z)\tan(\alpha))^2 = 0$.

latex(%, "d:/dynamics/precession/ConeEquation.tex")

Conical Coordinates



Transformation matrix from conical to Cartesian coordinates:

$$\begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix}$$

For later convenience, make this and its inverse into functions.

$M := \%$

$ConToCart := (a, b) \rightarrow \mathrm{evalm}(\mathrm{subs}(\alpha = a, \eta = b, \mathrm{eval}(M)))$

$CartToCon := (a, b) \rightarrow \mathrm{map}(simplify, \mathrm{inverse}(\mathrm{ConToCart}(a, b)))$

latex(ConToCart(α, η), "d:/dynamics/precession/ConicalToCartesian.tex")

latex(CartToCon(α, η), "d:/dynamics/precession/CartesianToConical.tex")

Hence,

$\mathrm{mat}(xhat, yhat, zhat) = \mathrm{ConToCart}(\alpha, \eta)\ \&*\ \mathrm{mat}(rhohat, etahat, alphahat)$

$$\begin{bmatrix} xhat \\ yhat \\ zhat \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} rhohat \\ etahat \\ alphahat \end{bmatrix}$$

and

mat(*rhohat*, *etahat*, *alphahat*) = CartToCon(α, η) &* mat(*xhat*, *yhat*, *zhat*)

$$\begin{bmatrix} rhohat \\ etahat \\ alphahat \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} xhat \\ yhat \\ zhat \end{bmatrix}$$

Radiation Pressure

Let $[P_X, P_Y, P_Z]$ be the pressure vector in the fixed frame. Then the conical coordinates representation of P is

mat(P_ρ, P_η, P_α) = (CartToCon(α, η) &* R(φ, ψ, θ)) &* mat(P_X, P_Y, P_Z)

$$\begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} = \left(\begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \right.$$

$$[\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi),\ \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi),\ \sin(\theta)\sin(\psi)]$$

$$[-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi),\ -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi),$$
$$\cos(\theta)\sin(\psi)]$$

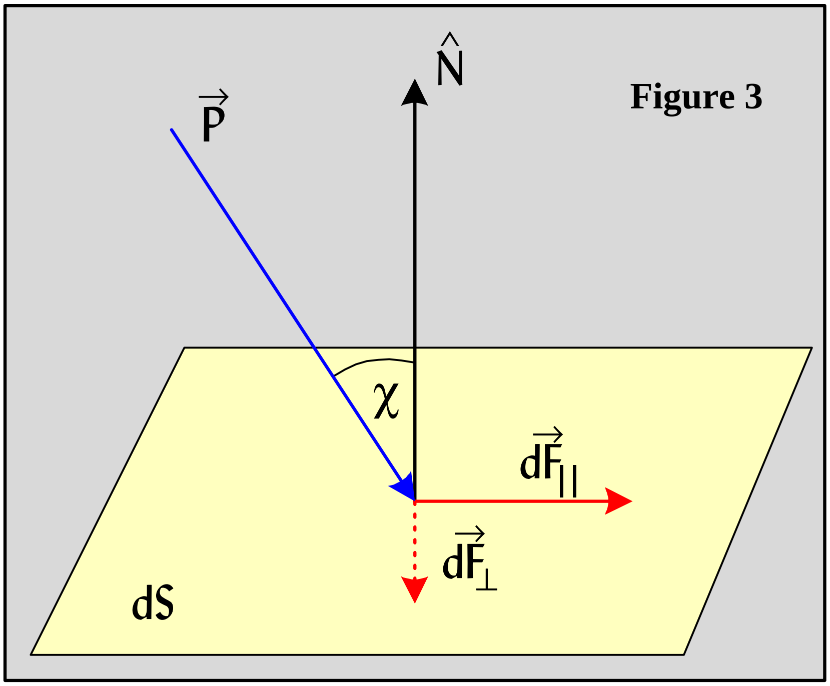
$$\left. [\sin(\psi)\sin(\phi),\ -\sin(\psi)\cos(\phi),\ \cos(\psi)] \right) \&* \begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix}$$

latex(%, "d:/dynamics/precession/PressureBodyFrame.tex")

Pbody := convert(evalm(rhs(%)), *vector*)

simplify(mag(%))

$$\sqrt{P_Z^2 + P_Y^2 + P_X^2}$$



The force on an area element *dS* is

$\mathrm{mat}(dF_{perp}, dF_{parallel}) = |P|\,|dS\,\cos(\gamma)|\,\mathrm{mat}((1+A_C)\cos(\gamma), (1-A_C)\sin(\gamma))$

$$\begin{bmatrix} dF_{perp} \\ dF_{parallel} \end{bmatrix} = |P|\,|dS|\,\cos(\gamma)|\begin{bmatrix} (1+A_C)\cos(\gamma) \\ (1-A_C)\sin(\gamma) \end{bmatrix}$$

where F_{perp} and $F_{parallel}$ are the component perpendicular and parallel to *dS*, γ is the angle between *P* and the (unit) surface normal *N*, and A_C is the albedo of the cone surface. From Figure 3, we see that $\mathrm{dot}(P, N) < 0$ for our problem. Hence, $\cos\gamma = -\cos\chi$ (that is, $\cos\chi = -\dfrac{\mathrm{dot}(P, N)}{|P|}$) and we have

$\mathrm{mat}(dF_{perp}, dF_{parallel}) = |P|\,dS\cos(\chi)\,\mathrm{mat}(-(1+A_C)\cos(\chi), (1-A_C)\sin(\chi))$

$$\begin{bmatrix} dF_{perp} \\ dF_{parallel} \end{bmatrix} = |P|\,dS\cos(\chi)\begin{bmatrix} -(1+A_C)\cos(\chi) \\ (1-A_C)\sin(\chi) \end{bmatrix}$$

The magnitude of *dF* is

$dFpp := \mathrm{convert}(\mathrm{evalm}(\mathrm{rhs}(\%)), vector)$

$\mathrm{simplify}(\sqrt{\mathrm{dot}(\%, \%)}, assume = real)$

$|dF| = \mathrm{rootfunc}(\%, collect, \cos, factor)$

$$|dF| = \mathrm{signum}(P)\,P\,\mathrm{signum}(dS)\,dS\,\mathrm{signum}(\cos(\chi))\cos(\chi)\sqrt{4\cos(\chi)^2 A_C + (A_C - 1)^2}$$

for p in select(has, indets(%), signum) do subs($p = 1$, %) od

subs($P = |P|$, %)

mag_dF := rhs(%)

$$|dF| = |P|\, dS \cos(\chi) \sqrt{4 \cos(\chi)^2 A_C + (A_C - 1)^2}$$

Now, $\cos\gamma = \frac{P_\alpha}{|P|}$. Hence, $\cos\chi = -\frac{P_\alpha}{|P|}$ or

$\cos\chi$ = subs($P_X = \pi_X$, $P_Y = \pi_Y$, $P_Z = \pi_Z$, $-Pbody_3$)

cos_chi := rhs(%)

latex(%, "d:/dynamics/precession/cos_chi.tex")

$$\cos\chi = -(\cos(\alpha)\cos(\eta)(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi)) + \cos(\alpha)\sin(\eta)(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi)) + \sin(\alpha)\sin(\psi)\sin(\phi))\,\pi_X - (\cos(\alpha)\cos(\eta)(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi)) + \cos(\alpha)\sin(\eta)(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi)) - \sin(\alpha)\sin(\psi)\cos(\phi))\,\pi_Y - (\cos(\alpha)\cos(\eta)\sin(\theta)\sin(\psi) + \cos(\alpha)\sin(\eta)\cos(\theta)\sin(\psi) + \sin(\alpha)\cos(\psi))\,\pi_Z$$

where $\pi_X = \frac{P_X}{|P|}$, $\pi_Y = \frac{P_Y}{|P|}$, $\pi_Z = \frac{P_Z}{|P|}$. $dF_{parallel}$ can be written $dF_{parallel} = \frac{|dF_{parallel}|(P - \mathrm{dot}(P, N)\,N)}{|P - \mathrm{dot}(P, N)\,N|}$. But $|P - \mathrm{dot}(P, N)\,N| = |\mathrm{cross}(P, N)|$ or $|P - \mathrm{dot}(P, N)\,N| = |P| \sin(\chi)$, and $\frac{\mathrm{dot}(P, N)}{|P|} = -\cos(\chi)$, so we can write

$\frac{dF_{parallel}}{|dF_{parallel}|} = \frac{P + |P|\cos(\chi)\,N}{|P|\sin(\chi)}$

$$\frac{dF_{parallel}}{|dF_{parallel}|} = \frac{P + |P|\cos(\chi)\,N}{|P|\sin(\chi)}$$

In component form,

subs($|P| = Q$, $P = \mathrm{mat}(P_\rho, P_\eta, P_\alpha)$, $Q = |P|$, $N = \mathrm{mat}(0, 0, 1)$, %)

$$\frac{dF_{parallel}}{|dF_{parallel}|} = \frac{\begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} + |P| \cos(\chi) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{|P| \sin(\chi)}$$

For dF_{perp}, we have $\frac{dF_{perp}}{|dF_{perp}|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore,

$\mathrm{mat}(dF_\rho, dF_\eta, dF_\alpha) = \mathrm{evalm}(dFpp_2\ \mathrm{rhs}(\%\%) + dFpp_1\ \mathrm{rhs}(\%))$

$$\begin{bmatrix} dF_\rho \\ dF_\eta \\ dF_\alpha \end{bmatrix} = \begin{bmatrix} dS \cos(\chi)\,(1 - A_C)\,P_\rho \\ dS \cos(\chi)\,(1 - A_C)\,P_\eta \\ dS \cos(\chi)\,(1 - A_C)\,(P_\alpha + |P| \cos(\chi)) - |P|\, dS \cos(\chi)^2\,(1 + A_C) \end{bmatrix}$$

The component perpendicular to the surface simplifies:

$\mathrm{factormat}(\mathrm{subs}(P_\rho = \pi_\rho |P|, P_\eta = \pi_\eta |P|, P_\alpha = \pi_\alpha |P|, \%\%), collect, [dS, \cos], factor)$

$$\frac{dF_{perp}}{|dF_{perp}|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the force integrated over the cone surface is

$\mathrm{mat}(F_\rho, F_\eta, F_\alpha) = \int_0^{2\pi} \int_f^{f+S} \mathrm{subs}(dS = \rho \sin(\alpha), \mathrm{rhs}(\%))\, d\rho\, d\eta$

$$\begin{bmatrix} F_\rho \\ F_\eta \\ F_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\rho\, d\eta$$

$Fintegral := \%$

latex(%, "d:/dynamics/precession/ConeForceIntegral.tex")

Torque Due to Radiation Pressure on the Cone Surface

Radius Vector to Cone Surface

To calculate the torque, we first need the radius vector to a point on the surface of the cone. The transformation from conical to cartesian body coordinates is $\left[x = \rho \sin(\alpha) \cos(\eta), y = \rho \sin(\alpha) \sin(\eta), z = h - \rho \cos(\alpha) + \frac{a}{\tan(\alpha)}\right]$.

$xyz := \%$

Hence, the radius vector from the center of mass to a point on the cone surface is, in the conical frame,

$\mathrm{subs}(xyz, \mathrm{CartToCon}(\alpha, \eta)\ \&*\ \mathrm{mat}(x, y, z))$

$\mathrm{map}(collect, \mathrm{evalm}(\%), [h, a], simplify)$

$r\_cone := \%$

$$\begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} \rho\sin(\alpha)\cos(\eta) \\ \rho\sin(\alpha)\sin(\eta) \\ h - \rho\cos(\alpha) + \dfrac{a}{\tan(\alpha)} \end{bmatrix}$$

$$\begin{bmatrix} -\cos(\alpha)\,h - \dfrac{\cos(\alpha)^2\,a}{\sin(\alpha)} + \rho \\ 0 \\ \sin(\alpha)\,h + \cos(\alpha)\,a \end{bmatrix}$$

latex(mat(*xyz*), "d:/dynamics/precession/ConicalCoordinates.tex")

latex(*r_cone*, "d:/dynamics/precession/r_cone.tex")

Statement of the Torque Integral in the Conical Frame

The torque integrated over the cone surface is therefore

$\mathrm{subs}(F = K, \mathrm{subsop}([2, 1, 1] = \text{'}cross\text{'}(\mathrm{eval}(r\_cone), \mathrm{op}([2, 1, 1], Fintegral)), Fintegral))$

$torque\_cone\_integral := \%$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \mathrm{cross}\left(\begin{bmatrix} -\cos(\alpha)\,h - \dfrac{\cos(\alpha)^2\,a}{\sin(\alpha)} + \rho \\ 0 \\ \sin(\alpha)\,h + \cos(\alpha)\,a \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) d\rho\, d\eta$$

which can be simplified as follows.

$\mathrm{op}([2, 1, 1], torque\_cone\_integral)$

$$\left[0, \cos(\alpha)\,h + \frac{\cos(\alpha)^2\,a}{\sin(\alpha)} - \rho, 0 \right]$$

$\mathrm{factormat}(\mathrm{mat}(\mathrm{subs}(A_C - 1 = -Q, \mathrm{eval}(\%))), collect, [Q, \pi_\alpha, \pi_\rho, \pi_\eta, \rho, A_C], factor)$

$$\left((\rho \sin(\alpha) - \cos(\alpha)(\sin(\alpha) h + \cos(\alpha) a)) \begin{bmatrix} 0 \\ \dfrac{\left(-\rho + \dfrac{\cos(\alpha)(\sin(\alpha) h + \cos(\alpha) a)}{\sin(\alpha)}\right)\sin(\alpha)}{\rho \sin(\alpha) - \cos(\alpha)(\sin(\alpha) h + \cos(\alpha) a)} \\ 0 \end{bmatrix} \right) / \sin(\alpha)$$

$\mathrm{subs}(\sin(\alpha)\, h + \cos(\alpha)\, a = U, \%)$

$\mathrm{factormat}(\%, collect, [Q, \pi_\alpha, \pi_\rho, A_C], factor)$

$$\frac{(\rho \sin(\alpha) - \cos(\alpha)\, U) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}{\sin(\alpha)}$$

$\mathrm{map}(x \to \mathrm{algsubs}(Q\, U = V, x), \mathrm{map}(expand, \mathrm{evalm}(\%)))$

$\mathrm{factormat}(\%, collect, [\rho, V], factor)$

$$\frac{(\rho \sin(\alpha) - \cos(\alpha)\, U) \begin{bmatrix} 0 \\ \dfrac{\left(-\rho + \dfrac{\cos(\alpha)\, U}{\sin(\alpha)}\right)\sin(\alpha)}{\rho \sin(\alpha) - \cos(\alpha)\, U} \\ 0 \end{bmatrix}}{\sin(\alpha)}$$

$arg := \%$

$subslist := [Q = 1 - A_C, U = \sin(\alpha)\, h + \cos(\alpha)\, a, V = Q\, U]$

$\mathrm{map}((x, y) \to \mathrm{lhs}(x) = \mathrm{subs}(y, \mathrm{rhs}(x)), subslist, subslist)$

$subslist := \%$

$$subslist := [Q = 1 - A_C, U = \sin(\alpha)\, h + \cos(\alpha)\, a, V = Q\, U]$$

Check:

$\mathrm{map}(simplify, \mathrm{evalm}(\mathrm{op}([2, 1, 1], torque\_cone\_integral) - \mathrm{subs}(subslist, arg)))$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, we have

$\mathrm{subsop}([2, 1, 1] = arg, torque\_cone\_integral)$

$torque\_cone\_integral := \%$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} \frac{(\rho \sin(\alpha) - \cos(\alpha)\, U) \begin{bmatrix} 0 \\ \dfrac{\left(-\rho + \dfrac{\cos(\alpha)\, U}{\sin(\alpha)}\right) \sin(\alpha)}{\rho \sin(\alpha) - \cos(\alpha)\, U} \\ 0 \end{bmatrix}}{\sin(\alpha)} \, d\rho \, d\eta$$

where

$\mathrm{mat}(subslist)$

$$\begin{bmatrix} Q = 1 - A_C \\ U = \sin(\alpha)\, h + \cos(\alpha)\, a \\ V = (1 - A_C)\, (\sin(\alpha)\, h + \cos(\alpha)\, a) \end{bmatrix}$$

and

$\cos(\chi) = cos\_chi$

$$\begin{aligned} \cos(\chi) = {} & -(\cos(\alpha) \cos(\eta) (\cos(\theta) \cos(\phi) - \sin(\theta) \cos(\psi) \sin(\phi)) \\ & + \cos(\alpha) \sin(\eta) (-\sin(\theta) \cos(\phi) - \cos(\theta) \cos(\psi) \sin(\phi)) + \sin(\alpha) \sin(\psi) \sin(\phi))\, \pi_X - (\\ & \cos(\alpha) \cos(\eta) (\cos(\theta) \sin(\phi) + \sin(\theta) \cos(\psi) \cos(\phi)) \\ & + \cos(\alpha) \sin(\eta) (-\sin(\theta) \sin(\phi) + \cos(\theta) \cos(\psi) \cos(\phi)) - \sin(\alpha) \sin(\psi) \cos(\phi))\, \pi_Y \\ & - (\cos(\alpha) \cos(\eta) \sin(\theta) \sin(\psi) + \cos(\alpha) \sin(\eta) \cos(\theta) \sin(\psi) + \sin(\alpha) \cos(\psi))\, \pi_Z \end{aligned}$$

Partial Evaluation of the Torque Integral in the Conical Frame

$torque\_cone\_integral$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} \frac{(\rho \sin(\alpha) - \cos(\alpha)\, U) \begin{bmatrix} 0 \\ \dfrac{\left(-\rho + \dfrac{\cos(\alpha)\, U}{\sin(\alpha)}\right) \sin(\alpha)}{\rho \sin(\alpha) - \cos(\alpha)\, U} \\ 0 \end{bmatrix}}{\sin(\alpha)} \, d\rho \, d\eta$$

We can do the integral over ρ right away. We note that $f = \frac{a}{\sin(\alpha)}$ and $S = \frac{b-a}{\sin(\alpha)}$. Hence,

$\mathrm{subs}\left(f + S = \frac{b}{\sin(\alpha)}, f = \frac{a}{\sin(\alpha)}, \%\right)$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_{\frac{a}{\sin(\alpha)}}^{\frac{b}{\sin(\alpha)}} \frac{(\rho \sin(\alpha) - \cos(\alpha)\, U) \begin{bmatrix} 0 \\ \dfrac{\left(-\rho + \dfrac{\cos(\alpha)\, U}{\sin(\alpha)}\right) \sin(\alpha)}{\rho \sin(\alpha) - \cos(\alpha)\, U} \\ 0 \end{bmatrix}}{\sin(\alpha)} \, d\rho \, d\eta$$

Evaluating the integral in ρ, we find

$\mathrm{map}\left(\mathit{int}, \mathrm{map}(\mathit{collect}, \mathrm{evalm}(\mathrm{op}([2, 1, 1], \mathrm{subs}(\pi_\rho = \mathit{Prho}, \mathit{torque\_cone\_integral}))), \rho), \rho = \frac{a}{\sin(\alpha)} .. \frac{b}{\sin(\alpha)}\right)$

$\mathrm{factormat}(\mathrm{subs}(\mathit{Prho} = \pi_\rho, V = Q\, U, \mathrm{map}(\mathit{expand}, \mathrm{evalm}(\%))), \mathit{collect}, [\cos(\chi), \pi_\rho, \pi_\eta, \pi_\alpha, A_C, Q, U], \mathit{factor})$

$\mathrm{subsop}([2, 1] = \%, \%\%\%)$

$\mathit{torque\_cone} := \%$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} -\frac{1}{2} \left((-b+a)(-a+2\cos(\alpha)U-b) \begin{bmatrix} 0 \\ -2\frac{\left(-\frac{\cos(\alpha)(-b+a)U}{\sin(\alpha)^2}+\frac{1}{2}\frac{(-b+a)(a+b)}{\sin(\alpha)^2}\right)\sin(\alpha)^2}{(-b+a)(-a+2\cos(\alpha)U-b)} \\ 0 \end{bmatrix} \right) \Big/ \sin(\alpha)^2 \, d\eta$$

Conversion of the Torque Integral to the Body Frame

Define $B_1 = \frac{(b+a)\cos(\alpha)U}{2\sin(\alpha)^2} - \frac{b^2+ba+a^2}{3\sin(\alpha)^2}$ and $B_2 = \frac{(b+a)UQ}{2\sin(\alpha)}$. Then the torque integral becomes

$$B_2 = \frac{1}{2}\frac{(a+b)UQ}{\sin(\alpha)}$$

$Bsubs := [\%\%, \%]$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} (-a+b)\cos(\chi)\left|P\right| \begin{bmatrix} -B_2\,\pi_\eta \\ -2\,B_1\,A_C\cos(\chi)+B_2\,\pi_\rho+B_1\,Q\,\pi_\alpha \\ -B_1\,Q\,\pi_\eta \end{bmatrix} d\eta$$

Check:

map(*simplify*, evalm(subs(*Bsubs*, op([2, 1], %)) − op([2, 1], *torque_cone*)))

$$\left[\frac{1}{2}\frac{\cos(\chi)\left|P\right|UQ\,\pi_\eta\,(-b^2+a^2)}{\sin(\alpha)}\right]$$

$$\left[\frac{1}{6}(-3\,b^2+3\,a^2+6\cos(\alpha)\,U\,b-6\cos(\alpha)\,U\,a+2\left|P\right|\cos(\chi)\,Q\,\pi_\alpha\,b^3\right.$$

$$- 6\left|P\right|\cos(\chi)^2 A_C \cos(\alpha)\, U a^2 + 4\left|P\right|\cos(\chi)^2 A_C a^3 + 3\left|P\right|\cos(\chi)\, U Q \pi_\rho \sin(\alpha)\, a^2$$

$$+ 3\left|P\right|\cos(\chi)\, Q \pi_\alpha \cos(\alpha)\, U a^2 - 2\left|P\right|\cos(\chi)\, Q \pi_\alpha a^3 - 4\left|P\right|\cos(\chi)^2 A_C b^3$$

$$+ 6\left|P\right|\cos(\chi)^2 A_C \cos(\alpha)\, U b^2 - 3\left|P\right|\cos(\chi)\, U Q \pi_\rho \sin(\alpha)\, b^2$$

$$- 3\left|P\right|\cos(\chi)\, Q \pi_\alpha \cos(\alpha)\, U b^2\Big) \Big/ \left(-1 + \cos(\alpha)^2\right)\Bigg]$$

$$\left[-\frac{1}{6}\frac{\cos(\chi)\left|P\right| Q \pi_\eta \left(-3\cos(\alpha)\, U b^2 + 2 b^3 + 3\cos(\alpha)\, U a^2 - 2 a^3\right)}{-1 + \cos(\alpha)^2}\right]$$

torque_cone := %%

latex(*torque_cone*, "d:/dynamics/precession/ConeTorqueIntegralCone.tex")

The pressure components in the cartesian body frame are

mat(π_x, π_y, π_z) = '*R*'(ϕ, ψ, θ) &* mat(π_X, π_Y, π_Z)

$$\begin{bmatrix} \pi_x \\ \pi_y \\ \pi_z \end{bmatrix} = \mathrm{R}(\phi, \psi, \theta)\ \&*\ \begin{bmatrix} \pi_X \\ \pi_Y \\ \pi_Z \end{bmatrix}$$

Similarly, the conversion from the cartesian body frame to the conical body frame is

mat($\pi_\rho, \pi_\eta, \pi_\alpha$) = CartToCon(α, η) &* mat(π_x, π_y, π_z)

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_\alpha \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} \pi_x \\ \pi_y \\ \pi_z \end{bmatrix}$$

Hence, we have

subsop([2, 2] = rhs('%%'), %)

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_\alpha \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \left(\mathrm{R}(\phi, \psi, \theta)\ \&*\ \begin{bmatrix} \pi_X \\ \pi_Y \\ \pi_Z \end{bmatrix}\right)$$

factormat(evalm(%), *collect*, [sin(η), cos(η)])

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_\alpha \end{bmatrix} =$$

$$[(\sin(\alpha)\,(-\sin(\theta)\cos(\phi)-\cos(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$+\sin(\alpha)\,(-\sin(\theta)\sin(\phi)+\cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y+\sin(\alpha)\cos(\theta)\sin(\psi)\,\pi_Z)\sin(\eta)$$
$$+(\sin(\alpha)\,(\cos(\theta)\cos(\phi)-\sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$+\sin(\alpha)\,(\cos(\theta)\sin(\phi)+\sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y+\sin(\alpha)\sin(\theta)\sin(\psi)\,\pi_Z)\cos(\eta)$$
$$-\pi_X\cos(\alpha)\sin(\psi)\sin(\phi)+\pi_Y\cos(\alpha)\sin(\psi)\cos(\phi)-\pi_Z\cos(\alpha)\cos(\psi)]$$
$$[((-\cos(\theta)\cos(\phi)+\sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$+(-\cos(\theta)\sin(\phi)-\sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y-\sin(\theta)\sin(\psi)\,\pi_Z)\sin(\eta)+($$
$$(-\sin(\theta)\cos(\phi)-\cos(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$+(-\sin(\theta)\sin(\phi)+\cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y+\pi_Z\cos(\theta)\sin(\psi))\cos(\eta)]$$
$$[(\cos(\alpha)\,(-\sin(\theta)\cos(\phi)-\cos(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$+\cos(\alpha)\,(-\sin(\theta)\sin(\phi)+\cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y+\cos(\alpha)\cos(\theta)\sin(\psi)\,\pi_Z)\sin(\eta)$$
$$+(\cos(\alpha)\,(\cos(\theta)\cos(\phi)-\sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$+\cos(\alpha)\,(\cos(\theta)\sin(\phi)+\sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y+\cos(\alpha)\sin(\theta)\sin(\psi)\,\pi_Z)\cos(\eta)$$
$$+\pi_X\sin(\alpha)\sin(\psi)\sin(\phi)-\pi_Y\sin(\alpha)\sin(\psi)\cos(\phi)+\pi_Z\sin(\alpha)\cos(\psi)]$$

$Psubs := [\mathrm{seq}(\mathrm{lhs}(\%)_{k,1}=\mathrm{rhs}(\%)_{k,1}, k=1..3)]$

We must transform the torque components from the conical frame to the body cartesian frame.

torque_cone

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} (b-a)\cos(\chi)\,|P| \begin{bmatrix} -B_2\,\pi_\eta \\ -2\,B_1\,A_C\cos(\chi)+B_2\,\pi_\rho+B_1\,Q\,\pi_\alpha \\ -B_1\,Q\,\pi_\eta \end{bmatrix} d\eta$$

$\mathrm{subsop}([2,1]=\mathrm{ConToCart}(\alpha,\eta)\ \&*\ \mathrm{op}([2,1], torque\_cone), torque\_cone)$

$\mathrm{subs}(K_\rho=K_x, K_\eta=K_y, K_\alpha=K_z, \%)$

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \&* $$

$$(b-a)\cos(\chi)\left|P\right| \begin{bmatrix} -B_2\,\pi_\eta \\ -2\,B_1\,A_C\cos(\chi) + B_2\,\pi_\rho + B_1\,Q\,\pi_\alpha \\ -B_1\,Q\,\pi_\eta \end{bmatrix} d\eta$$

subsop([2, 1] = factormat(evalm(op([2, 1], %)), *collect*, [π_ρ, π_η, π_α], *factor*), %)

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} (-b+a)\left|P\right|\cos(\chi)$$

$$[\sin(\eta)\,B_2\,\pi_\rho + \cos(\eta)\,(\sin(\alpha)\,B_2 + \cos(\alpha)\,B_1\,Q)\,\pi_\eta + \sin(\eta)\,B_1\,Q\,\pi_\alpha$$
$$-\,2\sin(\eta)\,B_1\,A_C\cos(\chi)]$$
$$[-\cos(\eta)\,B_2\,\pi_\rho + \sin(\eta)\,(\sin(\alpha)\,B_2 + \cos(\alpha)\,B_1\,Q)\,\pi_\eta - \cos(\eta)\,B_1\,Q\,\pi_\alpha$$
$$+\,2\cos(\eta)\,B_1\,A_C\cos(\chi)]$$
$$[(-\cos(\alpha)\,B_2 + \sin(\alpha)\,B_1\,Q)\,\pi_\eta]\,d\eta$$

foo := %

Define $C_1 = \sin(\alpha)\,B_2 + \cos(\alpha)\,B_1\,Q$ and $C_2 = -\sin(\alpha)\,Q\,B_1 + \cos(\alpha)\,B_2$. Then

Csubs := [%%, %]

invsubs(*Csubs*, *foo*)

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} (-b+a)\left|P\right|\cos(\chi)$$

$$\begin{bmatrix} \sin(\eta)\,B_2\,\pi_\rho + \cos(\eta)\,C_1\,\pi_\eta + \sin(\eta)\,B_1\,Q\,\pi_\alpha - 2\sin(\eta)\,B_1\,A_C\cos(\chi) \\ -\cos(\eta)\,B_2\,\pi_\rho + \sin(\eta)\,C_1\,\pi_\eta - \cos(\eta)\,B_1\,Q\,\pi_\alpha + 2\cos(\eta)\,B_1\,A_C\cos(\chi) \\ (-\cos(\alpha)\,B_2 + \sin(\alpha)\,B_1\,Q)\,\pi_\eta \end{bmatrix} d\eta$$

Finally, substitute for $\cos\chi$ and the pressure components and perform the integration.

$\mathrm{subs}(Psubs, \cos(\chi) = cos\_chi, \%)$

$\mathrm{map}(int, \mathrm{map}(collect, \mathrm{evalm}(\mathrm{op}([2, 1], \%)), [\sin(\eta), \cos(\eta), |P|]), \eta)$

$foo := \%$

for k **to** 3 **do** $foo_{k,1} := \mathrm{subs}(\eta = 2\pi, foo_{k,1}) - \mathrm{subs}(\eta = 0, foo_{k,1}); foo_{k,1} := \mathrm{factor}(foo_{k,1})$

od

$\mathrm{factormat}(foo, collect, [\pi_X, \pi_Y, \pi_Z])$

$\mathrm{mat}(K_x, K_y, K_z) = \%$

$torque\_xyz\_cone := \%$

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = (-b + a)\,\pi\,|P|\,(-\sin(\psi)\,\pi_Y \cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)$$

$$(2\sin(\alpha)\,B_1\,Q\cos(\alpha) + \sin(\alpha)^2\,B_2 + 4\sin(\alpha)\,B_1\,A_C\cos(\alpha) + C_1\sin(\alpha) - B_2\cos(\alpha)^2)$$

$$[(\cos(\theta)\cos(\psi)\sin(\phi) + \sin(\theta)\cos(\phi))\,\pi_X + (\sin(\theta)\sin(\phi) - \cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y$$
$$- \pi_Z\cos(\theta)\sin(\psi)]$$

$$[(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y$$
$$+ \sin(\theta)\sin(\psi)\,\pi_Z]$$

$$[0]$$

$\mathrm{latex}(torque\_xyz\_cone, \text{"d:/dynamics/precession/ConeTorqueBody.tex"})$

We can simplify C_1 and C_2 somewhat:

$\mathrm{subs}(\sin(\alpha)^2 + \cos(\alpha)^2 = 1, \mathrm{collect}(\mathrm{subs}(Bsubs, Csubs_1), [Q, U], factor))$

$\mathrm{subs}(subslist_1, \mathrm{collect}(\%, [\sin(\alpha)]))$

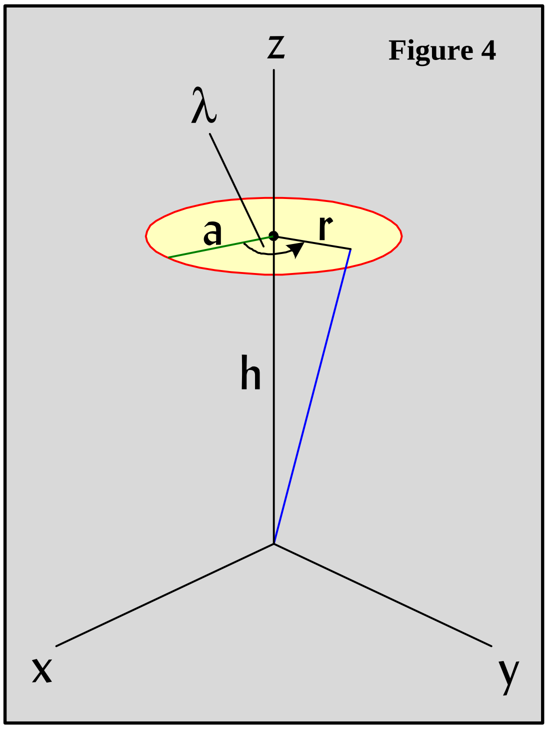
$$C_1 = \frac{\left(\frac{1}{2}(a+b)\,U - \frac{1}{3}\cos(\alpha)\,(b^2 + b\,a + a^2)\right)(1 - A_C)}{\sin(\alpha)^2}$$

$\mathrm{subs}(subslist_1, \mathrm{factor}(\mathrm{subs}(Bsubs, Csubs_2)))$

$$C_2 = \frac{1}{3}\,\frac{(1 - A_C)\,(b^2 + b\,a + a^2)}{\sin(\alpha)}$$

$Csubs := [\%\%, \%]$

Torque Due to Radiation Pressure on the Flattop Surface



$$\mathrm{subs}(Csubs, Bsubs, subslist, A_C = A_T, a = 0, b = a, torque\_xyz\_cone)\Big|_{\alpha = \frac{\pi}{2}}$$

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = -a^2 \pi |P| (-\sin(\psi)\,\pi_Y \cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\, h\,(1 - A_T)$$

$$[(\cos(\theta)\cos(\psi)\sin(\phi) + \sin(\theta)\cos(\phi))\,\pi_X + (\sin(\theta)\sin(\phi) - \cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y - \pi_Z \cos(\theta)\sin(\psi)]$$

$$[(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y + \sin(\theta)\sin(\psi)\,\pi_Z]$$

$$[0]$$

$torque\_xyz\_top := \%$

latex($torque\_xyz\_top$, "d:/dynamics/precession/FlatTopTorqueBody.tex")

The Equations of Motion

The equations of motion are

mat($FirstOrderODEsK$)

$$\left[\frac{\partial}{\partial t}\phi = \Omega_{\phi}\right]$$

$$\left[\frac{\partial}{\partial t}\psi = \Omega_{\psi}\right]$$

$$\left[\frac{\partial}{\partial t}\theta = \Omega_{\theta}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right) = ((1-\beta)\,\Omega_{\theta} - \cos(\psi)\,(1+\beta)\,\Omega_{\phi})\,\Omega_{\psi} + \frac{\cos(\theta)\,K_y + K_x\sin(\theta)}{I_{xy}}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_{\psi} = \left(\cos(\psi)\,\beta\,\Omega_{\phi}^{\,2} + (-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi) + \frac{-K_y\sin(\theta) + \cos(\theta)\,K_x}{I_{xy}}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_{\phi} + (-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi} + \frac{-\frac{\sin(\psi)\,K_z}{-1+\beta} - (\cos(\theta)\,K_y + K_x\sin(\theta))\cos(\psi)}{I_{xy}}\right]$$

Torque Combinations

Assemble the torque combinations.

$Kcone$:= convert(evalm(rhs($torque\_xyz\_cone$)), $vector$)

$Ktop$:= convert(evalm(rhs($torque\_xyz\_top$)), $vector$)

$(Kcone_1 + Ktop_1)\sin(\theta) + (Kcone_2 + Ktop_2)\cos(\theta)$

applyrule($\sin(\theta)^2 + \cos(\theta)^2 = 1$, factor(%))

map($collect$, %, $[B_1, B_2, C_1]$, $factor$)

$torque_{\phi}$:= %

$$\pi\,|P|\,(\pi_X\cos(\phi) + \sin(\phi)\,\pi_Y)\,(-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,(2\cos(\alpha)\sin(\alpha)\,(2A_C + Q)\,(-b+a)\,B_1 - (\cos(\alpha) - \sin(\alpha))\,(\cos(\alpha) + \sin(\alpha))\,(-b+a)\,B_2 + \sin(\alpha)\,(-b+a)\,C_1 + a^2\,h\,(-1 + A_T))$$

and

$(Kcone_1 + Ktop_1)\cos(\theta) - (Kcone_2 + Ktop_2)\sin(\theta)$

$\mathrm{applyrule}(\sin(\theta)^2 + \cos(\theta)^2 = 1, \mathrm{factor}(\%))$

$\mathrm{map}(collect, \%, [B_1, B_2, C_1], factor)$

$torque_\psi := \%$

$$-|P|\pi(-\sin(\psi)\pi_Y\cos(\phi) + \cos(\psi)\pi_Z + \sin(\psi)\sin(\phi)\pi_X)$$
$$(\sin(\psi)\pi_Z - \sin(\phi)\pi_X\cos(\psi) + \cos(\phi)\cos(\psi)\pi_Y)($$
$$2\cos(\alpha)\sin(\alpha)(2A_C + Q)(-b+a)B_1$$
$$-(\cos(\alpha) - \sin(\alpha))(\cos(\alpha) + \sin(\alpha))(-b+a)B_2 + \sin(\alpha)(-b+a)C_1$$
$$+ a^2 h(-1 + A_T))$$

where

Csubs

$$\left[C_1 = \frac{\left(\frac{1}{2}(a+b)U - \frac{1}{3}\cos(\alpha)(b^2 + ba + a^2)\right)(1 - A_C)}{\sin(\alpha)^2}, C_2 = \frac{1}{3}\frac{(1 - A_C)(b^2 + ba + a^2)}{\sin(\alpha)}\right]$$

Bsubs

$$\left[B_1 = \frac{1}{2}\frac{(a+b)\cos(\alpha)U}{\sin(\alpha)^2} - \frac{1}{3}\frac{b^2 + ba + a^2}{\sin(\alpha)^2}, B_2 = \frac{1}{2}\frac{(a+b)UQ}{\sin(\alpha)}\right]$$

subslist

$$[Q = 1 - A_C, U = \sin(\alpha)h + \cos(\alpha)a, V = (1 - A_C)(\sin(\alpha)h + \cos(\alpha)a)]$$

Simplification of the Torque Combinations

Let's check the commonality between the two torque combinations.

$\mathrm{mat}(K_x\sin(\theta) + K_y\cos(\theta), K_x\cos(\theta) - K_y\sin(\theta)) =$

$\mathrm{factormat}(\mathrm{subs}(subslist_1, \mathrm{mat}(torque_\phi, torque_\psi)), collect, [B_1, B_2, C_1], factor)$

$$\begin{bmatrix} \cos(\theta)\, K_y + K_x \sin(\theta) \\ -K_y \sin(\theta) + \cos(\theta)\, K_x \end{bmatrix} = \pi \left| P \right| \left(-\sin(\psi)\, \pi_Y \cos(\phi) + \cos(\psi)\, \pi_Z + \sin(\psi) \sin(\phi)\, \pi_X\right) \Big(2 \cos(\alpha) \sin(\alpha) (1 + A_C) (-b + a) B_1 - (\cos(\alpha) - \sin(\alpha)) (\cos(\alpha) + \sin(\alpha)) (-b + a) B_2 + \sin(\alpha) (-b + a) C_1 + a^2 h (-1 + A_T)\Big) \begin{bmatrix} \pi_X \cos(\phi) + \sin(\phi)\, \pi_Y \\ -\sin(\psi)\, \pi_Z + \sin(\phi)\, \pi_X \cos(\psi) - \cos(\phi) \cos(\psi)\, \pi_Y \end{bmatrix}$$

torque_terms := %

select(*has*, rhs(%), *a*)

$$2 \cos(\alpha) \sin(\alpha) (1 + A_C) (-b + a) B_1 - (\cos(\alpha) - \sin(\alpha)) (\cos(\alpha) + \sin(\alpha)) (-b + a) B_2 + \sin(\alpha) (-b + a) C_1 + a^2 h (-1 + A_T)$$

L := location(%%, %)

$$L := [2, 4]$$

subs(*Csubs*, *Bsubs*, $subslist_1$, %%)

$$2 \cos(\alpha) \sin(\alpha) (1 + A_C) (-b + a) \left(\frac{1}{2} \frac{(a + b) \cos(\alpha)\, U}{\sin(\alpha)^2} - \frac{1}{3} \frac{b^2 + b\, a + a^2}{\sin(\alpha)^2} \right) - \frac{1}{2} \frac{(\cos(\alpha) - \sin(\alpha)) (\cos(\alpha) + \sin(\alpha)) (-b + a) (a + b)\, U (1 - A_C)}{\sin(\alpha)} + \frac{(-b + a) \left(\frac{1}{2} (a + b)\, U - \frac{1}{3} \cos(\alpha) (b^2 + b\, a + a^2) \right) (1 - A_C)}{\sin(\alpha)} + a^2 h (-1 + A_T)$$

bar := %

latex(%, "d:/dynamics/precession/torque_term_to_simplify.tex")

Collect(%, $[b - a, b + a, b^2 + b\, a + a^2, U, A_C]$, *simplify*, *loc*)

$$\left(\frac{(-2 \cos(\alpha)^2 b + 2\, a \cos(\alpha)^2 + b - a)\, A_C}{\sin(\alpha)} + \frac{-b + a}{\sin(\alpha)} \right) U (a + b) + \left(-\frac{1}{3} \frac{\cos(\alpha) (-b + a)\, A_C}{\sin(\alpha)} - \frac{\cos(\alpha) (-b + a)}{\sin(\alpha)} \right) (b^2 + b\, a + a^2) + a^2 h (-1 + A_T)$$

location(%, remove(*has*, select(*has*, %, *U*), *b* − *a*))

$$[1]$$

subsop(% = map(*factor*, op(%, %%)), %%)

$$\frac{(2 A_C \cos(\alpha)^2 - A_C + 1)(-b + a) U (a + b)}{\sin(\alpha)} + \left(-\frac{1}{3} \frac{\cos(\alpha)(-b + a) A_C}{\sin(\alpha)} - \frac{\cos(\alpha)(-b + a)}{\sin(\alpha)} \right) (b^2 + b a + a^2) + a^2 h (-1 + A_T)$$

latex(%, "d:/dynamics/precession/torque_term_simplified.tex")

Check:

simplify(% − *bar*)

$$0$$

subs(*subslist*, subsop(L = %%, *torque_terms*))

$$\begin{bmatrix} \cos(\theta) K_y + K_x \sin(\theta) \\ -K_y \sin(\theta) + \cos(\theta) K_x \end{bmatrix} = \pi \left| P \right| (-\sin(\psi) \pi_Y \cos(\phi) + \cos(\psi) \pi_Z + \sin(\psi) \sin(\phi) \pi_X) \left(\frac{(2 A_C \cos(\alpha)^2 - A_C + 1)(-b + a)(\sin(\alpha) h + \cos(\alpha) a)(a + b)}{\sin(\alpha)} + \left(-\frac{1}{3} \frac{\cos(\alpha)(-b + a) A_C}{\sin(\alpha)} - \frac{\cos(\alpha)(-b + a)}{\sin(\alpha)} \right) (b^2 + b a + a^2) + a^2 h (-1 + A_T) \right) \begin{bmatrix} \pi_X \cos(\phi) + \sin(\phi) \pi_Y \\ -\sin(\psi) \pi_Z + \sin(\phi) \pi_X \cos(\psi) - \cos(\phi) \cos(\psi) \pi_Y \end{bmatrix}$$

torque_terms := %

This looks good. Now define

select(*has*, rhs(*torque_terms*), ψ)

$$(-\sin(\psi) \pi_Y \cos(\phi) + \cos(\psi) \pi_Z + \sin(\psi) \sin(\phi) \pi_X) \begin{bmatrix} \pi_X \cos(\phi) + \sin(\phi) \pi_Y \\ -\sin(\psi) \pi_Z + \sin(\phi) \pi_X \cos(\psi) - \cos(\phi) \cos(\psi) \pi_Y \end{bmatrix}$$

select(*has*, %, $\sin(\psi) \pi_Z$)

$$\begin{bmatrix} \pi_X \cos(\phi) + \sin(\phi) \pi_Y \\ -\sin(\psi) \pi_Z + \sin(\phi) \pi_X \cos(\psi) - \cos(\phi) \cos(\psi) \pi_Y \end{bmatrix}$$

gsubs := [$g_0(\phi, \psi)$ = remove(*has*, %%, $\sin(\psi) \pi_Z$), $g_1(\phi, \psi) = \%_{1, 1}$, $g_2(\phi, \psi) = \%_{2, 1}$,

$G(a, b, h, A_C, A_T, \alpha) = \mathrm{select}(has, \mathrm{remove}(has, \mathrm{rhs}(torque\_terms), \psi), \{a, P, \pi\})]$

Then we can write

$\mathrm{invsubs}(gsubs, \mathrm{remove}(has, \mathrm{rhs}(torque\_terms), \psi))$

$$G(a, b, h, A_C, A_T, \alpha)$$

$\mathrm{invsubs}(gsubs, \mathrm{select}(has, \mathrm{rhs}(torque\_terms), \psi))$

$$g_0(\phi, \psi) \begin{bmatrix} g_1(\phi, \psi) \\ g_2(\phi, \psi) \end{bmatrix}$$

$\mathrm{lhs}(torque\_terms) = \%\ \%\%$

$$\begin{bmatrix} \cos(\theta) K_y + K_x \sin(\theta) \\ -K_y \sin(\theta) + \cos(\theta) K_x \end{bmatrix} = g_0(\phi, \psi) \begin{bmatrix} g_1(\phi, \psi) \\ g_2(\phi, \psi) \end{bmatrix} G(a, b, h, A_C, A_T, \alpha)$$

$torque\_terms\_subs := \%$

where

$\mathrm{mat}(gsubs)$

$$[g_0(\phi, \psi) = -\sin(\psi) \pi_Y \cos(\phi) + \cos(\psi) \pi_Z + \sin(\psi) \sin(\phi) \pi_X]$$

$$[g_1(\phi, \psi) = \pi_X \cos(\phi) + \sin(\phi) \pi_Y]$$

$$[g_2(\phi, \psi) = -\sin(\psi) \pi_Z + \sin(\phi) \pi_X \cos(\psi) - \cos(\phi) \cos(\psi) \pi_Y]$$

$$\Bigg[G(a, b, h, A_C, A_T, \alpha) = \pi \left|P\right| \Bigg(\frac{(2 A_C \cos(\alpha)^2 - A_C + 1)(-b + a)(\sin(\alpha) h + \cos(\alpha) a)(a + b)}{\sin(\alpha)} + \left(-\frac{1}{3} \frac{\cos(\alpha)(-b + a) A_C}{\sin(\alpha)} - \frac{\cos(\alpha)(-b + a)}{\sin(\alpha)} \right) (b^2 + b a + a^2) + a^2 h (-1 + A_T) \Bigg) \Bigg]$$

Equations of Motion

$\mathrm{subs}(\mathrm{seq}(\mathrm{lhs}(torque\_terms\_subs)_{k,1} = \mathrm{evalm}(\mathrm{rhs}(torque\_terms\_subs))_{k,1}, k = 1 .. 2), K_z = 0, FirstOrderODEsK)$

$\mathrm{mat}(\%)$

$$\left[\frac{\partial}{\partial t} \phi = \Omega_\phi \right]$$

$$\left[\frac{\partial}{\partial t}\psi = \Omega_{\psi}\right]$$

$$\left[\frac{\partial}{\partial t}\theta = \Omega_{\theta}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right) = ((1-\beta)\,\Omega_{\theta} - \cos(\psi)\,(1+\beta)\,\Omega_{\phi})\,\Omega_{\psi} + \frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_1(\phi,\psi)}{I_{xy}}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_{\psi} = \left(\cos(\psi)\,\beta\,\Omega_{\phi}^{\;2} + (-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi) + \frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_2(\phi,\psi)}{I_{xy}}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_{\phi} + (-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi} - \frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_1(\phi,\psi)\cos(\psi)}{I_{xy}}\right]$$

FirstOrderODEs := %

where

mat(*gsubs*)

$$[g_0(\phi,\psi) = -\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X]$$

$$[g_1(\phi,\psi) = \pi_X\cos(\phi) + \sin(\phi)\,\pi_Y]$$

$$[g_2(\phi,\psi) = -\sin(\psi)\,\pi_Z + \sin(\phi)\,\pi_X\cos(\psi) - \cos(\phi)\cos(\psi)\,\pi_Y]$$

$$\left[\mathrm{G}(a,b,h,A_C,A_T,\alpha) = \pi\,|P|\left(\frac{(2A_C\cos(\alpha)^2 - A_C + 1)\,(-b+a)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(a+b)}{\sin(\alpha)} + \left(-\frac{1}{3}\,\frac{\cos(\alpha)\,(-b+a)\,A_C}{\sin(\alpha)} - \frac{\cos(\alpha)\,(-b+a)}{\sin(\alpha)}\right)(b^2 + b\,a + a^2) + a^2\,h\,(-1+A_T)\right)\right]$$

latex(*FirstOrderODEs*, "d:/dynamics/precession/FirstOrderODEs.tex");

latex(mat($gsubs$), "d:/dynamics/precession/gsubs.tex")

Fast Spin Approximation

The equations of motion are

$eqs :=$
$\mathrm{subs}(\mathrm{op}(\mathrm{select}(has, gsubs, \psi)), \mathrm{G}(a, b, h, A_C, A_T, \alpha) = G, [\mathrm{seq}(FirstOrderODEs_{k,1}, k = 4 .. 6)])$

mat(%)

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{(-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G\,(\pi_X\cos(\phi) + \sin(\phi)\,\pi_Y)}{I_{xy}}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_\psi = \left(\cos(\psi)\,\beta\,\Omega_\phi^{\ 2} + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) + \left((-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G\,(-\sin(\psi)\,\pi_Z + \sin(\phi)\,\pi_X\cos(\psi) - \cos(\phi)\cos(\psi)\,\pi_Y)\right) \Big/ I_{xy}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi - \frac{(-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G\,(\pi_X\cos(\phi) + \sin(\phi)\,\pi_Y)\cos(\psi)}{I_{xy}}\right]$$

For purposes of substitution, which we'll use in a moment, divide by $\sin(\psi)$ where appropriate.

$\mathrm{select}\left(has, eqs, \frac{\partial}{\partial t}\Omega_\phi\right)$

$L := \mathrm{location}(eqs, \mathrm{op}(\%))$

$\mathrm{subsop}\left(\% = \frac{\mathrm{op}(\%\%)}{\sin(\psi)}, eqs\right)$

$\mathrm{select}\left(has, \%, \frac{\partial}{\partial t}\Omega_\theta\right)$

$L := \mathrm{location}(\%\%, \mathrm{op}(\%))$

$\mathrm{subsop}\left(\% = \frac{\mathrm{op}(\%\%)}{\sin(\psi)}, \%\%\%\right)$

$odesubs := \%$

mat(%)

$$\left[\frac{\partial}{\partial t}\Omega_{\phi} = \left(((1-\beta)\,\Omega_{\theta} - \cos(\psi)\,(1+\beta)\,\Omega_{\phi})\,\Omega_{\psi} + \frac{(-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G\,(\pi_X\cos(\phi) + \sin(\phi)\,\pi_Y)}{I_{xy}}\right) / \sin(\psi)\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_{\psi} = \left(\cos(\psi)\,\beta\,\Omega_{\phi}^{\ 2} + (-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi) + \left((-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G\,(-\sin(\psi)\,\pi_Z + \sin(\phi)\,\pi_X\cos(\psi) - \cos(\phi)\cos(\psi)\,\pi_Y)\right) \Big/ I_{xy}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_{\theta} = \left(((\cos(\psi)^2\,\beta + 1)\,\Omega_{\phi} + (-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi} - \frac{(-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G\,(\pi_X\cos(\phi) + \sin(\phi)\,\pi_Y)\cos(\psi)}{I_{xy}}\right) / \sin(\psi)\right]$$

Differentiate the equations of motion.

map(*diff*, *eqs*, *t*)

eqs2 := %

mat(%)

$$\left[\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right) + \sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_{\phi}\right) = \left((1-\beta)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right) + \sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)(1+\beta)\,\Omega_{\phi} - \cos(\psi)\,(1+\beta)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right)\right)\Omega_{\psi} + ((1-\beta)\,\Omega_{\theta} - \cos(\psi)\,(1+\beta)\,\Omega_{\phi})\left(\frac{\partial}{\partial t}\Omega_{\psi}\right) + \left(\left(-\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Y\cos(\phi) + \sin(\psi)\,\pi_Y\sin(\phi)\left(\frac{\partial}{\partial t}\phi\right) - \sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Z + \cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\sin(\phi)\,\pi_X + \sin(\psi)\cos(\phi)\left(\frac{\partial}{\partial t}\phi\right)\pi_X\right)G\,(\pi_X\cos(\phi) + \sin(\phi)\,\pi_Y)\right) \Big/ I_{xy} + \Big((-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,G$$

$$\left(-\pi_X \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) + \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right) \pi_Y \right)\Big)\Big) \Big/ I_{xy} \Big]$$

$$\Big[\frac{\partial^2}{\partial t^2} \Omega_\psi = \Big(-\sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) \beta\, \Omega_\phi^{\,2} + 2\cos(\psi)\, \beta\, \Omega_\phi \left(\frac{\partial}{\partial t} \Omega_\phi\right) + (-1+\beta) \left(\frac{\partial}{\partial t} \Omega_\theta\right) \Omega_\phi$$

$$+ (-1+\beta)\, \Omega_\theta \left(\frac{\partial}{\partial t} \Omega_\phi\right)\Big) \sin(\psi) + \left(\cos(\psi)\, \beta\, \Omega_\phi^{\,2} + (-1+\beta)\, \Omega_\theta\, \Omega_\phi\right) \cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) + \Big(\Big($$

$$-\cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \pi_Y \cos(\phi) + \sin(\psi)\, \pi_Y \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) - \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) \pi_Z$$

$$+ \cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \sin(\phi)\, \pi_X + \sin(\psi) \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right) \pi_X \Big) G$$

$$(-\sin(\psi)\, \pi_Z + \sin(\phi)\, \pi_X \cos(\psi) - \cos(\phi) \cos(\psi)\, \pi_Y)\Big) \Big/ I_{xy} + \Big($$

$$(-\sin(\psi)\, \pi_Y \cos(\phi) + \cos(\psi)\, \pi_Z + \sin(\psi) \sin(\phi)\, \pi_X)\, G \Big(-\cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \pi_Z$$

$$+ \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right) \pi_X \cos(\psi) - \sin(\phi)\, \pi_X \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) + \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) \cos(\psi)\, \pi_Y$$

$$+ \cos(\phi) \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) \pi_Y \Big)\Big) \Big/ I_{xy} \Big]$$

$$\Big[\cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \left(\frac{\partial}{\partial t} \Omega_\theta\right) + \sin(\psi) \left(\frac{\partial^2}{\partial t^2} \Omega_\theta\right) = \Big(-2\cos(\psi)\, \beta \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) \Omega_\phi$$

$$+ (\cos(\psi)^2 \beta + 1) \left(\frac{\partial}{\partial t} \Omega_\phi\right) - (-1+\beta) \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) \Omega_\theta + (-1+\beta) \cos(\psi) \left(\frac{\partial}{\partial t} \Omega_\theta\right)\Big) \Omega_\psi$$

$$+ ((\cos(\psi)^2 \beta + 1)\, \Omega_\phi + (-1+\beta) \cos(\psi)\, \Omega_\theta) \left(\frac{\partial}{\partial t} \Omega_\psi\right) - \Big(\Big(-\cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \pi_Y \cos(\phi)$$

$$+ \sin(\psi)\, \pi_Y \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) - \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right) \pi_Z + \cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \sin(\phi)\, \pi_X$$

$$+ \sin(\psi) \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right) \pi_X \Big) G\, (\pi_X \cos(\phi) + \sin(\phi)\, \pi_Y) \cos(\psi)\Big) \Big/ I_{xy} - \Big($$

$$(-\sin(\psi)\, \pi_Y \cos(\phi) + \cos(\psi)\, \pi_Z + \sin(\psi) \sin(\phi)\, \pi_X)\, G$$

$$\left(-\pi_X \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) + \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right) \pi_Y \right) \cos(\psi)\Big) \Big/ I_{xy} + \Big($$

$$(-\sin(\psi)\, \pi_Y \cos(\phi) + \cos(\psi)\, \pi_Z + \sin(\psi) \sin(\phi)\, \pi_X)\, G\, (\pi_X \cos(\phi) + \sin(\phi)\, \pi_Y) \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right)$$

$$\Big) \Big/ I_{xy} \Big]$$

Substitute for $\frac{\partial}{\partial t}\Omega_\theta$, $\frac{\partial}{\partial t}\Omega_\phi$, and $\frac{\partial}{\partial t}\Omega_\psi$ using the original equations of motion. Then, assume Ω_θ is large and the pressure terms (i.e., $\mathrm{G}(a, b, h, A_C, A_T, \alpha)$) are small. We find the resulting second-order differential equations,

```
subsODEs := proc(ode, odesubs)
local q, Lq, u, Lu, eqn;
    eqn := copy(ode);
    if not (isdiff(eqn) and 1 < difforder(eqn)) then for q in eqn do
            Lq := location(eqn, q);
            if isdiff(q) and difforder(q) = 1 then q := subs(odesubs, q)
            elif 1 < nops(q) and not isdiff(u) then q := procname(q, odesubs)
            fi;
            eqn := subsop(Lq = q, eqn)
        od
    fi;
    eval(eqn)
end
```

Check:

```
for ii to 3 do factor(subs(odesubs, rhs(eqs2[ii])) - subsODEs(rhs(eqs2[ii]), odesubs)) od
```

$$0$$

$$0$$

$$0$$

```
subsODEs(eqs2, odesubs)
```

$\mathrm{subs}\left(\frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\phi = \Omega_\phi, \%\right)$

```
convert(expansion(%, [Ω_ψ, Ω_ϕ, G], 1), diff)
Collect(%, [G], factor)
eqs2 := invsubs([op(gsubs), lhs(op(select(has, gsubs, G))) = G], %)
mat(%)
```

$$\left[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\phi\right) = -\frac{(-1+\beta)\,\Omega_\theta\, g_0(\phi,\psi)\, g_2(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)}{I_{xy}} - (-1+\beta)^2\,\Omega_\theta^{\,2}\sin(\psi)\,\Omega_\phi\right]$$

$$\left[\frac{\partial^2}{\partial t^2}\Omega_\psi = \frac{(-1+\beta)\,\Omega_\theta\, g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\, g_1(\phi,\psi)}{I_{xy}} - (-1+\beta)^2\,\Omega_\theta^{\,2}\,\Omega_\psi\right]$$

$$\left[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\theta\right) = (-1+\beta)^2\cos(\psi)\,\Omega_\theta^{\;2}\sin(\psi)\,\Omega_\phi + \frac{(-1+\beta)\cos(\psi)\,\Omega_\theta\,g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_2(\phi,\psi)}{I_{xy}}\right]$$

Since Ω_θ is large, we may also assume its rate of change is small, hence $\frac{\partial^2}{\partial t^2}\Omega_\theta = 0$. The first consequence of this is that the angular velocity of the inclination, Ω_ψ, exhibits simple harmonic motion: the solution of

$$\left(\frac{\partial^2}{\partial t^2}\Omega_\psi\right) + (1-\beta)^2\,\Omega_\theta^{\;2}\,\Omega_\psi + \frac{(1-\beta)\,g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_1(\phi,\psi)\,\Omega_\theta}{I_{xy}} = 0$$

is just

$\mathrm{collect}(\mathrm{subs}(\omega = \Omega_\theta, g_0 = g_0(\phi,\psi), g_1 = g_1(\phi,\psi),$
$\mathrm{dsolve}(\mathrm{subs}(\Omega_\theta = \omega, g_1(\phi,\psi) = g_1, g_0(\phi,\psi) = g_0, \%), \Omega_\psi)), [\_C1, \_C2], \mathit{factor})$

$$\Omega_\psi = \frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_1(\phi,\psi)}{\Omega_\theta\,(-1+\beta)\,I_{xy}} + \_C1\sin((-1+\beta)\,\Omega_\theta\,t) + \_C2\cos((-1+\beta)\,\Omega_\theta\,t)$$

which is simple harmonic motion with a small linear drift in $\psi(t)$. The second consequence stems from either the first or the third equation (both say the same thing). Setting $\frac{\partial^2}{\partial t^2}\Omega_\theta = 0$ places a constraint on the value of the longitudinal motion — i.e., the precession rate. We then have

$\mathrm{subs}(\mathrm{select}(\mathit{has}, \mathit{gsubs}, g), \mathrm{isolate}(\mathrm{rhs}(\mathit{eqs2}_3), \Omega_\phi))$

$\mathit{precession\_rate} := \%$

$$\Omega_\phi = -\big((-\sin(\psi)\,\pi_Y\cos(\phi) + \cos(\psi)\,\pi_Z + \sin(\psi)\sin(\phi)\,\pi_X)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,(-\sin(\psi)\,\pi_Z + \sin(\phi)\,\pi_X\cos(\psi) - \cos(\phi)\cos(\psi)\,\pi_Y)\big) \big/ \big((-1+\beta)\,\Omega_\theta\,I_{xy}\sin(\psi)\big)$$

To make this expression a little bit clearer, further assume that the pressure is mainly along the Z axis — that is, that π_X and π_Y are small. Then this expression becomes

$$\mathrm{Collect}\left(\mathrm{expansion}(\%, [\pi_X, \pi_Y], 1), \left[\mathrm{G}(a,b,h,A_C,A_T,\alpha), \pi, |P|, \frac{1}{\Omega_\theta}, \frac{1}{I_{xy}}, -1+\beta, \pi_Z\right], \mathit{factor}\right)$$

$$\Omega_{\phi} = \left(\left(\cos(\psi)\,\pi_Z^{\,2} - \frac{(\cos(\psi)-\sin(\psi))\,(\cos(\psi)+\sin(\psi))\,(-\pi_Y\cos(\phi)+\pi_X\sin(\phi))\,\pi_Z}{\sin(\psi)}\right) \mathrm{G}(a, b, h, A_C, A_T, \alpha)\right) \Big/ \left((-1+\beta)\, I_{xy}\, \Omega_{\theta}\right)$$

Thus, we have a weak phase dependence on the non-axial pressure components. For the purely axial case $\pi_Z = 1$, we have

subs($\pi_Z = -1$, $\pi_X = 0$, $\pi_Y = 0$, %)

$$\Omega_{\phi} = \frac{\cos(\psi)\,\mathrm{G}(a, b, h, A_C, A_T, \alpha)}{(-1+\beta)\, I_{xy}\, \Omega_{\theta}}$$

Finally, the precession for a flat disk of uniform albedo, $\alpha = \frac{\pi}{2}$, $A_C = A$, and $A_T = A$, is

factor$\left(\text{eval}\left(\text{subs}\left(gsubs, \alpha = \frac{\pi}{2}, A_C = A, A_T = A, \%\right)\right)\right)$

$$\Omega_{\phi} = \frac{\cos(\psi)\,\pi\,|P|\,h\,b^2\,(-1+A)}{(-1+\beta)\,\Omega_{\theta}\, I_{xy}}$$

Let's look at the precession period $T_{\phi} = \frac{2\,\pi}{\Omega_{\phi}}$ as a function of cone angle and albedo. For a pressure field along the Z axis, we have

subs($\pi_Z = -1$, $\pi_X = 0$, $\pi_Y = 0$, *precession_rate*)

$T_{\phi} = \frac{2\,\pi}{\mathrm{rhs}(\%)}$

precession_period := %

$$T_{\phi} = 2\,\frac{\pi\,(-1+\beta)\, I_{xy}\, \Omega_{\theta}}{\cos(\psi)\,\mathrm{G}(a, b, h, A_C, A_T, \alpha)}$$

3D Plot of Precession Period

subs(select(*has*, *gsubs*, *G*), *precession_period*)

subs$\left(\psi = inclin, |P| = P, A_C = AC, A_T = AT, I_{xy} = Ixy, \Omega_{\theta} = \frac{2\,\pi}{Tspin}, \mathrm{rhs}(\%)\right)$

optimize(%, *tryhard*)

Tphi := makeproc([%], *parameters* = [α, *a*, *b*, *h*, *AC*, *AT*, *inclin*, *Tspin*, β, *Ixy*, *P*])

Tmin := −100.0

$Tmax := 100.0$

$ACmin := .7$

$ACmax := .95$

$Ncontours := 30$

$$Clevels := \left[\mathrm{seq}\left(Tmin + \frac{(ii - 1.0)\,(Tmax - Tmin)}{Ncontours - 1.0}, ii = 1 \,..\, Ncontours \right) \right]$$

$ConColors := [yellow, yellow]$

$alphamin := 84$

$alphamax := 92$

$ConPos := Tmin$

$orient := [40, 60]$

$$surfplot := \mathrm{plot3d}\left(\frac{\mathrm{Tphi}(\alpha\,deg, 1, 3, 1, AC, .8, 45\,deg, 2\,3600, .1, 200.0, 4.5\,10^{(-6)})}{3600.0}, \right.$$
$$AC = ACmin \,..\, ACmax, \alpha = alphamin \,..\, alphamax, grid = [60, 60], style = patchcontour,$$
$$\left. contours = Clevels \right)$$

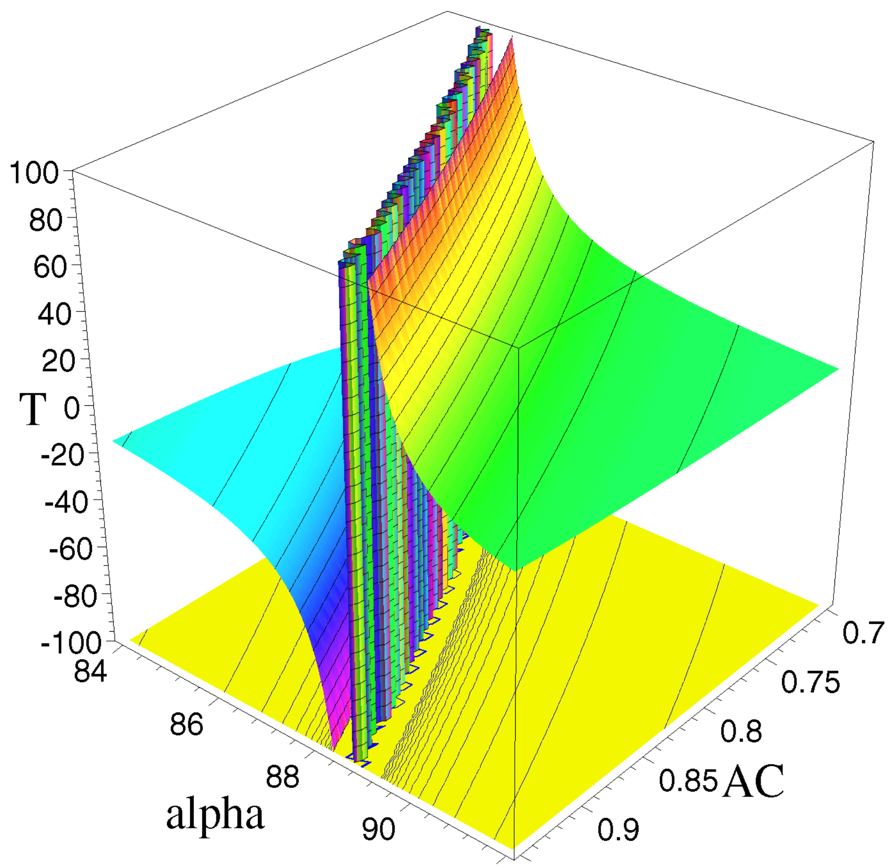
$$conplot := plots_{contourplot}\left(\frac{\mathrm{Tphi}(\alpha\,deg, 1, 3, 1, AC, .8, 45\,deg, 2\,3600, .1, 200.0, 4.5\,10^{(-6)})}{3600.0}, \right.$$
$$AC = ACmin \,..\, ACmax, \alpha = alphamin \,..\, alphamax, grid = [40, 40], contours = Clevels,$$
$$\left. coloring = ConColors, filled = true, view = Tmin \,..\, Tmax \right)$$

$$plot\_xform := plottools_{transform}((x, y) \rightarrow [x, y, ConPos])$$

$$disp := plots_{display}(\{surfplot, \mathrm{plot\_xform}(conplot)\}, view = Tmin \,..\, Tmax, projection = .9,$$
$$orientation = orient, labels = [\text{"AC"}, \text{"alpha"}, T],$$
$$title = \text{"Precession Period vs. Cone Angle and Albedo"})$$

$disp$

Precession Period vs. Cone Angle and Albedo



plotsetup(*gif*, *plotoutput* = "d:/dynamics/precession/PrecessionPeriod.gif", *plotoptions* = "width=1800,height=1800")

disp

plotsetup(*default*)

Precession Null

Numerical

We can determine the approximate angles α at which the precession is zero. From

precession_rate

$$\Omega_\phi = -\left((-\sin(\psi)\, \pi_Y \cos(\phi) + \cos(\psi)\, \pi_Z + \sin(\psi) \sin(\phi)\, \pi_X)\, G(a, b, h, A_C, A_T, \alpha) \, (-\sin(\psi)\, \pi_Z + \sin(\phi)\, \pi_X \cos(\psi) - \cos(\phi) \cos(\psi)\, \pi_Y) \right) \Big/ \left((-1+\beta)\, \Omega_\theta\, I_{xy} \sin(\psi) \right)$$

we see that for $\Omega_\phi = 0$ we must have $G(a, b, h, A_C, A_T, \alpha) = 0$. Hence, we require

$$\frac{\mathrm{rhs}(\mathrm{op}(\mathrm{select}(has, gsubs, G)))}{\pi |P|} = 0$$

$$\frac{(2 A_C \cos(\alpha)^2 - A_C + 1)(-b + a)(\sin(\alpha) h + \cos(\alpha) a)(a + b)}{\sin(\alpha)}$$
$$+ \left(-\frac{1}{3} \frac{\cos(\alpha)(-b + a) A_C}{\sin(\alpha)} - \frac{\cos(\alpha)(-b + a)}{\sin(\alpha)}\right)(b^2 + b a + a^2) + a^2 h (-1 + A_T) = 0$$

latex(%, "d:/dynamics/precession/PrecessionNullEq.tex")

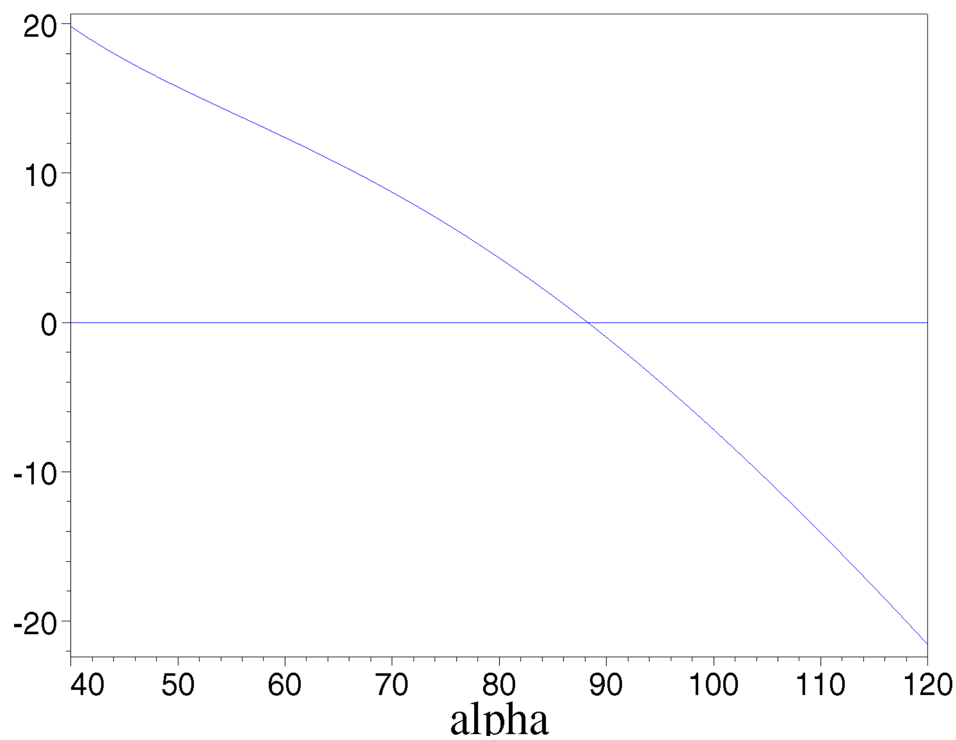
Plots

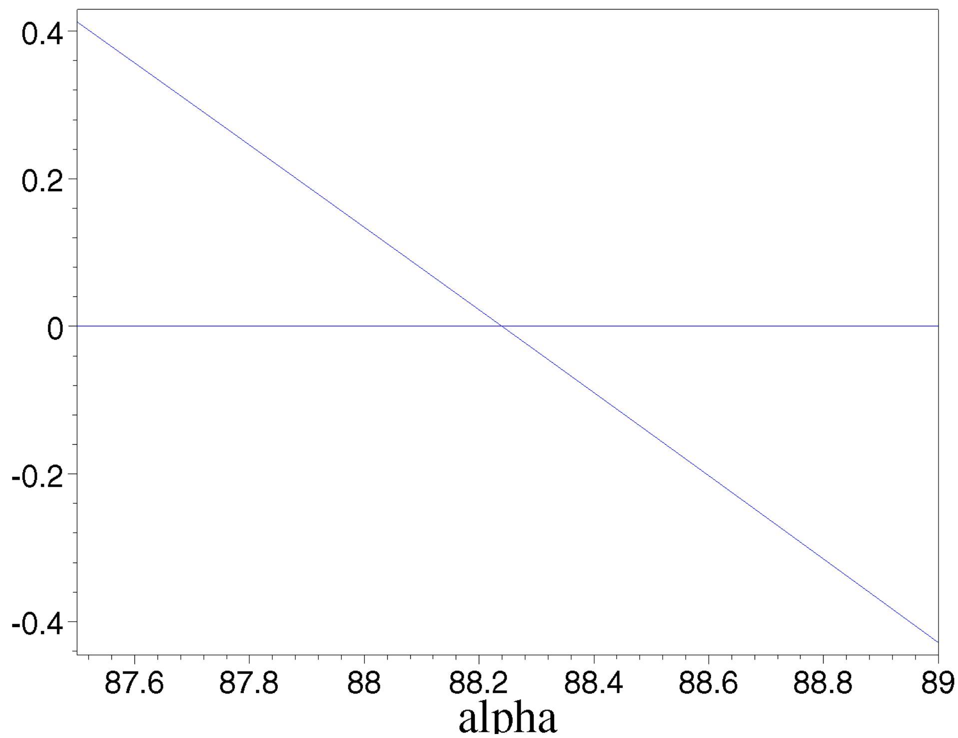
Here are plots of this function:

$NullEq := \mathrm{fn}(\mathrm{lhs}(\%), \alpha, a, b, h, A_C, A_T)$

plot([0, '*NullEq*'(α *deg*, 1, 3, 1, .9, .8)], α = 40 .. 120)

plot([0, '*NullEq*'(α *deg*, 1, 3, 1, .9, .8)], α = 87.5 .. 89)





The zero crossing, for this particular set of parameters, occurs at

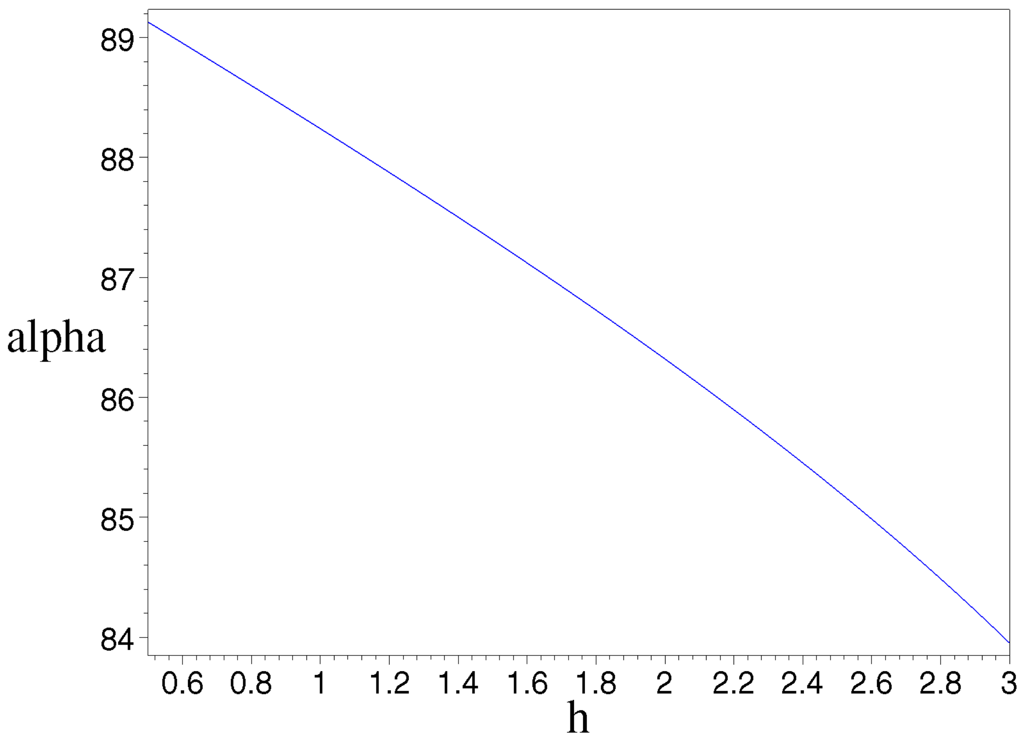
$\alpha = \text{fsolve}('NullEq'(\alpha\, deg, 1, 3, 1, .9, .8), \alpha, 80 .. 90)$

$$\alpha = 88.24000978$$

Here is a plot of zero-crossing angle as a function of h.

$Null := \textbf{proc}(a, b, h, AC, AT)\ \text{fsolve}(\text{NullEq}(\alpha * deg, a, b, h, AC, AT), \alpha, 30 .. 150)\ \textbf{end}$

$\text{plot}('Null'(1, 3, h, .9, .8), h = .5 .. 3, labels = [\text{"h"}, \text{"alpha"}], thickness = 2)$



plotsetup(*gif*, *plotoutput* = "d:/dynamics/precession/NullAngles.gif", *plotoptions* = "width=1000,height=1000")

plot('*Null*'(1, 3, *h*, .9, .8), *h* = .5 .. 3, *labels* = ["h", "alpha"])

plotsetup(*default*)

$\_N := 100$

$\_pts := \mathrm{array}(1 .. \_N, 1 .. 2)$

for k **to** $\_N$ **do** $\_h := \frac{(k - 1.0)\, 2.5}{\_N - 1.0} + .5;\ \_pts_{k,\,1} := \_h;\ \_pts_{k,\,2} := \mathrm{Null}(1, 3, \_h, .9, .8)$
od

writedata("d:/dynamics/precession/NullAngles.dat", *_pts*)

?

Analytic

The exact solutions of the zero-precession equation are arctangents of sixth-order polynomials — not a pretty sight. Let us instead try an iterative approach. We see that there is at minimum one useful solution, near $\alpha = \frac{\pi}{2}$. Try a solution of the form $\alpha = \frac{\pi}{2} + x_1\,\varepsilon + x_2\,\varepsilon^2 + x_3\,\varepsilon^3$. Plugging into the equation, we find

$\mathrm{Collect}(\mathrm{NullEq}(\alpha, a, b, h, A_C, A_T)\sin(\alpha), [b + a, -a + b], factor, loc)$

$$(2A_C\cos(\alpha)^2 - A_C + 1)(-b+a)(\sin(\alpha)h + \cos(\alpha)a)(a+b) + \frac{1}{3}\cos(\alpha)b^3 A_C + \cos(\alpha)b^3 - \frac{1}{3}\cos(\alpha)a^3 A_C - \cos(\alpha)a^3 - a^2 h\sin(\alpha) + a^2 h\sin(\alpha)A_T$$

collect(%, [sin, cos, h, A_C], *factor*)

$$(2A_C(-b+a)h(a+b)\cos(\alpha)^2 + (-A_C(-b+a)(a+b) - b^2 + a^2 A_T)h)\sin(\alpha) + 2A_C(-b+a)a(a+b)\cos(\alpha)^3 + \left(-\frac{1}{3}(-b+a)(b+2a)^2 A_C - b^2(-b+a)\right)\cos(\alpha)$$

subs(cos(α) = χ, sin(α) = $\sqrt{1-\chi^2}$, %)

isolate(%, $\sqrt{1-\chi^2}$)

$$\sqrt{1-\chi^2} = \frac{-2A_C(-b+a)a(a+b)\chi^3 - \left(-\frac{1}{3}(-b+a)(b+2a)^2 A_C - b^2(-b+a)\right)\chi}{2A_C(-b+a)h(a+b)\chi^2 + (-A_C(-b+a)(a+b) - b^2 + a^2 A_T)h}$$

map($x \to x^2$, %)

denom(rhs(%)) (lhs(%) − numer(rhs(%))) = 0

$$9h^2(-2A_C\chi^2 b^2 + 2A_C a^2\chi^2 + A_C b^2 - A_C a^2 - b^2 + a^2 A_T)^2\left(1-\chi^2 - \chi^2(-b+a)^2(6A_C a^2\chi^2 + 6A_C a\chi^2 b - A_C b^2 - 4A_C b a - 4A_C a^2 - 3b^2)^2\right) = 0$$

collect(%, [χ, h, A_C, A_T], *factor*)

$$-1296(-b+a)^4(a+b)^4 a^2 A_C^4 h^2\chi^{10} + \Big(432a(b^2+7ba+7a^2)(a+b)^3(-b+a)^4 A_C^4 + (-1296a^4(-b+a)^3(a+b)^3 A_T + 1296b^2 a(2a-b)(-b+a)^3(a+b)^3)A_C^3\Big) h^2\chi^8 + \Big(-36(b^4+20b^3a+93b^2a^2+146a^3b+73a^4)(a+b)^2(-b+a)^4 A_C^4 + ($$

$$216\,a^3\,(2\,b^2+11\,b\,a+11\,a^2)\,(a+b)^2\,(-b+a)^3\,A_T$$

$$-216\,b^2\,(-b^3-7\,b^2\,a+11\,b\,a^2+21\,a^3)\,(a+b)^2\,(-b+a)^3)\,A_C{}^3+\Big($$

$$-324\,a^6\,(-b+a)^2\,(a+b)^2\,A_T{}^2+648\,b^2\,a^3\,(3\,a-2\,b)\,(-b+a)^2\,(a+b)^2\,A_T$$

$$-36\,(54\,b^4\,a^2-54\,b^5\,a+1+9\,b^6)\,(-b+a)^2\,(a+b)^2\Big)A_C{}^2\Big)h^2\,\chi^6+\Big($$

$$36\,(b^2+7\,b\,a+7\,a^2)\,(a+b)^2\,(b+2\,a)^2\,(-b+a)^4\,A_C{}^4+($$

$$-36\,a^2\,(a+b)\,(b^2+10\,b\,a+10\,a^2)\,(b+2\,a)^2\,(-b+a)^3\,A_T$$

$$+36\,b^2\,(a+b)\,(-5\,b^4-19\,b^3\,a+27\,b^2\,a^2+113\,a^3\,b+73\,a^4)\,(-b+a)^3)\,A_C{}^3+\Big($$

$$108\,a^5\,(-b+a)^2\,(b+2\,a)^2\,(a+b)\,A_T{}^2$$

$$-216\,b^2\,a^2\,(a+b)\,(-b^3-5\,b^2\,a+4\,b\,a^2+11\,a^3)\,(-b+a)^2\,A_T$$

$$+36\,(a+b)\,(63\,a^3\,b^4+3\,a^2\,b^5-42\,a\,b^6+2\,a+3\,b^7+2\,b)\,(-b+a)^2\Big)A_C{}^2+\Big($$

$$324\,a^5\,(-b+a)^2\,b^2\,(a+b)\,A_T{}^2$$

$$-36\,a^2\,(a+b)\,(-b+a)\,(27\,b^4\,a^2-36\,b^5\,a+1+9\,b^6)\,A_T$$

$$+36\,b^2\,(-b+a)\,(a+b)\,(18\,b^4\,a^2-27\,b^5\,a+1+9\,b^6)\Big)A_C\Big)h^2\,\chi^4+\Big($$

$$-9\,(-b+a)^4\,(a+b)^2\,(b+2\,a)^4\,A_C{}^4+(18\,a^2\,(-b+a)^3\,(a+b)\,(b+2\,a)^4\,A_T$$

$$-18\,b^2\,(a+b)\,(-2\,b^2+4\,b\,a+7\,a^2)\,(b+2\,a)^2\,(-b+a)^3)\,A_C{}^3+\Big($$

$$-9\,a^4\,(-b+a)^2\,(b+2\,a)^4\,A_T{}^2$$

$$+18\,b^2\,a^2\,(-5\,b^2+4\,b\,a+10\,a^2)\,(b+2\,a)^2\,(-b+a)^2\,A_T$$

$$-9\,(73\,b^4\,a^4+80\,b^5\,a^3-30\,b^6\,a^2+5\,a^2-40\,b^7\,a+10\,b\,a-2\,b^8+5\,b^2)\,(-b+a)^2$$

$$\Big)A_C{}^2+\Big(-54\,a^4\,(-b+a)^2\,b^2\,(b+2\,a)^2\,A_T{}^2$$

$$+54\,a^2\,(-b+a)\,(11\,a^3\,b^4-3\,a^2\,b^5+a-9\,a\,b^6+b^7+b)\,A_T$$

$$-54\,b^2\,(-b+a)\,(7\,a^3\,b^4-3\,a^2\,b^5+a-6\,a\,b^6+2\,b^7+b)\Big)A_C$$

$$-9\,a^4\,(1+9\,b^6-18\,b^5\,a+9\,b^4\,a^2)\,A_T^{\;2}$$

$$+18\,b^2\,a^2\,(1+9\,b^6-18\,b^5\,a+9\,b^4\,a^2)\,A_T-9\,b^4\,(1+9\,b^6-18\,b^5\,a+9\,b^4\,a^2)\Big)h^2$$

$$\chi^2+\Big(9\,(-b+a)^2\,(a+b)^2\,A_C^{\;2}$$

$$+(-18\,a^2\,(-b+a)\,(a+b)\,A_T+18\,b^2\,(-b+a)\,(a+b))\,A_C+9\,a^4\,A_T^{\;2}-18\,b^2\,a^2\,A_T$$

$$+9\,b^4\Big)h^2=0$$

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$$\mathrm{subs}\left(\alpha=\frac{\pi}{2}+x_1\,\varepsilon+x_2\,\varepsilon^2+x_3\,\varepsilon^3,\ \mathrm{NullEq}(\alpha,a,b,h,A_C,A_T)\right)$$

$$\mathrm{expansion}(\%,\varepsilon,3)$$

$$NullEqX := \%$$

$$\Bigg(\Bigg((1-A_C)\,(-b+a)\left(-x_2\,x_1\,h-\left(x_3-\frac{1}{6}\,x_1^{\;3}\right)a\right)$$

$$+(4\,A_C\,x_2\,x_1\,(-b+a)-(A_C-1)\,(-b+a)\,x_2\,x_1)\,h$$

$$-\left(2\,A_C\,x_1^{\;2}\,(-b+a)-\frac{1}{2}\,(A_C-1)\,(-b+a)\,x_1^{\;2}\right)x_1\,a\Bigg)(a+b)$$

$$+\left(\frac{1}{3}\left(x_3+\frac{1}{3}\,x_1^{\;3}\right)(-b+a)\,A_C+\left(x_3+\frac{1}{3}\,x_1^{\;3}\right)(-b+a)\right)(b^2+b\,a+a^2)\Bigg)\varepsilon^3+\Bigg(\Bigg($$

$$(1-A_C)\,(-b+a)\left(-\frac{1}{2}\,x_1^{\;2}\,h-x_2\,a\right)$$

$$+\left(2\,A_C\,x_1^{\;2}\,(-b+a)-\frac{1}{2}\,(A_C-1)\,(-b+a)\,x_1^{\;2}\right)h\Bigg)(a+b)$$

$$+\left(\frac{1}{3}\,x_2\,(-b+a)\,A_C+x_2\,(-b+a)\right)(b^2+b\,a+a^2)\Bigg)\varepsilon^2+$$

$$\left(-(1-A_C)\,(-b+a)\,x_1\,a\,(a+b)+\left(\frac{1}{3}\,x_1\,(-b+a)\,A_C+x_1\,(-b+a)\right)(b^2+b\,a+a^2)\right)$$

$$\varepsilon+(1-A_C)\,(-b+a)\,h\,(a+b)+a^2\,h\,(-1+A_T)$$

$$\mathrm{Collect}(\mathrm{isolate}(\mathrm{subs}(\varepsilon^3=0,\varepsilon^2=0,\varepsilon=1,\%),x_1),[h,b],\mathit{factor})$$

$$x_1 = \frac{\left(\left(1-A_C\right) b^2 + a^2 \left(A_C - A_T\right)\right) h}{\left(-\frac{1}{3} A_C - 1\right) b^3 - \left(A_C - 1\right) a\, b^2 + \frac{4}{3} a^3 A_C}$$

location$\left(\%, \text{select}\left(has, \text{rhs}(\%), \frac{4\, A_C\, b\, a}{3}\right)\right)$

$$[1, 1]$$

collect(op(%, %%), $[A_C]$, *factor*)

$$1$$

subsop(%% = %, %%%)

Null_x1 := %

$$x_1 = \frac{\left(\left(1-A_C\right) b^2 + a^2 \left(A_C - A_T\right)\right) h}{\left(-\frac{1}{3} A_C - 1\right) b^3 - \left(A_C - 1\right) a\, b^2 + \frac{4}{3} a^3 A_C}$$

collect(isolate(subs($\varepsilon^3 = 0, \varepsilon^2 = 1, \varepsilon = 0$, *NullEqX*), x_2), $[h, A_C]$, *factor*)

$$x_2 = \frac{\left(-\left(-1 + 2\, x_1^{\,2}\right) (-b + a)\, (a + b)\, A_C + b^2 - a^2 A_T\right) h}{\frac{1}{3} (-b + a)\, (b + 2\, a)^2 A_C + b^2 (-b + a)}$$

subs(*Null_x1*, %)

Null_x2 := %

$$x_2 = \left(\left(-\left(-1 + 2 \frac{\left(\left(1-A_C\right) b^2 + a^2 \left(A_C - A_T\right)\right)^2 h^2}{\left(\left(-\frac{1}{3} A_C - 1\right) b^3 - \left(A_C - 1\right) a\, b^2 + \frac{4}{3} a^3 A_C\right)^2}\right) (-b + a)\, (a + b)\, A_C + b^2 - a^2 A_T\right) h\right) \Big/ \left(\frac{1}{3} (-b + a)\, (b + 2\, a)^2 A_C + b^2 (-b + a)\right)$$

Collect$\left(\text{isolate}(\text{subs}(\varepsilon^3 = 1, \varepsilon = 0, NullEqX), x_3), \left[h, x_1^{\,3}, A_C\right], factor, loc\right)$

$$x_3 = \Bigg(-4 \frac{x_2 (-b+a)(a+b) A_C x_1}{\frac{1}{3}(-b+a)(b+2a)^2 A_C + b^2 (-b+a)} + \frac{A_C (-b+a)(a+b) + b^2 - a^2 A_T}{\frac{1}{3}(-b+a)(b+2a)^2 A_C + b^2 (-b+a)} \Bigg) h + \frac{\left(\frac{1}{9}(-b+a)(-b^2+14\,b\,a+14\,a^2) A_C - \frac{1}{3} b^2 (-b+a)\right) x_1^3}{\frac{1}{3}(-b+a)(b+2a)^2 A_C + b^2 (-b+a)}$$

subs(*Null_x1*, *Null_x2*, *%*)

Null_x3 := *%*

$$x_3 = \Bigg(-4 \Bigg(\Bigg(\Bigg(-\left(-1 + 2 \frac{\left((1-A_C) b^2 + a^2 (A_C - A_T)\right)^2 h^2}{\left(\left(-\frac{1}{3}A_C - 1\right) b^3 - (A_C - 1)\, a\, b^2 + \frac{4}{3} a^3 A_C\right)^2} \right) (-b+a)(a+b) A_C + b^2$$
$$- a^2 A_T \Bigg) h^2 (-b+a)(a+b) A_C \left((1-A_C) b^2 + a^2 (A_C - A_T)\right) \Bigg) \Big/ \Bigg($$
$$\left(\frac{1}{3}(-b+a)(b+2a)^2 A_C + b^2 (-b+a)\right)^2$$
$$\left(\left(-\frac{1}{3}A_C - 1\right) b^3 - (A_C - 1)\, a\, b^2 + \frac{4}{3} a^3 A_C\right)\Bigg)\Bigg)$$
$$+ \frac{A_C (-b+a)(a+b) + b^2 - a^2 A_T}{\frac{1}{3}(-b+a)(b+2a)^2 A_C + b^2 (-b+a)} \Bigg) h + \Bigg($$
$$\left(\frac{1}{9}(-b+a)(-b^2 + 14\,b\,a + 14\,a^2) A_C - \frac{1}{3} b^2 (-b+a)\right)$$

$$((1-A_C)\,b^2+a^2\,(A_C-A_T))^3\,h^3\Big) \Big/ \left(\left(\frac{1}{3}(-b+a)\,(b+2\,a)^2\,A_C+b^2\,(-b+a)\right)\left(\left(-\frac{1}{3}A_C-1\right)b^3-(A_C-1)\,a\,b^2+\frac{4}{3}a^3\,A_C\right)^3\right)$$

latex(*Null_x1*, "d:/dynamics/precession/PrecessionNull_x1.tex")

latex(*Null_x2*, "d:/dynamics/precession/PrecessionNull_x2.tex")

latex(*Null_x3*, "d:/dynamics/precession/PrecessionNull_x3.tex")

Hence, we have, to second order, the result

$$\alpha=\frac{\pi}{2}+\mathrm{rhs}(Null\_x1)+\mathrm{rhs}(Null\_x2)+\mathrm{rhs}(Null\_x3)$$

$$\alpha=\frac{1}{2}\pi+\frac{((1-A_C)\,b^2+a^2\,(A_C-A_T))\,h}{\left(-\frac{1}{3}A_C-1\right)b^3-(A_C-1)\,a\,b^2+\frac{4}{3}a^3\,A_C}+\Bigg(\Bigg(\Bigg(-\left(-1+2\frac{((1-A_C)\,b^2+a^2\,(A_C-A_T))^2\,h^2}{\left(\left(-\frac{1}{3}A_C-1\right)b^3-(A_C-1)\,a\,b^2+\frac{4}{3}a^3\,A_C\right)^2}\right)(-b+a)\,(a+b)\,A_C+b^2$$

$$-a^2\,A_T\Bigg)h\Bigg)\Big/\left(\frac{1}{3}(-b+a)\,(b+2\,a)^2\,A_C+b^2\,(-b+a)\right)+\Bigg(-4\Bigg(\Bigg(\Bigg($$

$$-\left(-1+2\frac{((1-A_C)\,b^2+a^2\,(A_C-A_T))^2\,h^2}{\left(\left(-\frac{1}{3}A_C-1\right)b^3-(A_C-1)\,a\,b^2+\frac{4}{3}a^3\,A_C\right)^2}\right)(-b+a)\,(a+b)\,A_C+b^2$$

$$-a^2\,A_T\Bigg)h^2\,(-b+a)\,(a+b)\,A_C\,((1-A_C)\,b^2+a^2\,(A_C-A_T))\Bigg)\Big/\Bigg($$

$$\left(\frac{1}{3}(-b+a)\,(b+2\,a)^2\,A_C+b^2\,(-b+a)\right)^2$$

$$\left(\left(-\frac{1}{3}A_C-1\right)b^3-(A_C-1)\,a\,b^2+\frac{4}{3}a^3\,A_C\right)\Bigg)$$

$$+\frac{A_C(-b+a)(a+b)+b^2-a^2A_T}{\frac{1}{3}(-b+a)(b+2a)^2A_C+b^2(-b+a)}\Bigg)h+\Bigg(\left(\frac{1}{9}(-b+a)(-b^2+14ba+14a^2)A_C-\frac{1}{3}b^2(-b+a)\right)\left((1-A_C)b^2+a^2(A_C-A_T)\right)^3h^3\Bigg)\Bigg/\left(\left(\frac{1}{3}(-b+a)(b+2a)^2A_C+b^2(-b+a)\right)\left(\left(-\frac{1}{3}A_C-1\right)b^3-(A_C-1)ab^2+\frac{4}{3}a^3A_C\right)^3\right)$$

$ApproxNull := \mathrm{fn}\left(\frac{\mathrm{rhs}(\%)}{deg}, a, b, h, A_C, A_T\right)$

Null(1, 3, 1, .9, .8)

88.24000978

evalf(ApproxNull(1, 3, 1, .9, .8))

84.72164585

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save/restart

save "d:/dynamics/precession/precession.m"

restart

alias($I = I$)

alias($\psi = \psi(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$, $\Omega_x = \Omega_x(t)$, $\Omega_y = \Omega_y(t)$, $\Omega_z = \Omega_z(t)$, $\Omega_\phi = \Omega_\phi(t)$, $\Omega_\psi = \Omega_\psi(t)$, $\Omega_\theta = \Omega_\theta(t)$)

read "d:/dynamics/precession/precession.m"

Id, fn, det, size, rows, cols, transpose, inverse, augment, stack, extend, grad, curl, div, laplacian, angle, intparts, ψ, θ, ϕ, Ω_x, Ω_y, Ω_z, Ω_ϕ, Ω_ψ, Ω_θ

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