STRETCHING OF A CABLE WITH A WEIGHT ATTACHED

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1. SOLID CORE CABLE

Consider a solid steel cable of radius R with a weight of mass m attached. Now drop this mass a distance L, then apply brakes to the top of the cable, stopping the mass. How much will the cable stretch?

Assuming we don't snap the cable (i.e., that we're in the linear regime), we can characterize it as a spring, with spring constant k. Hence, near equilibrium we have simple harmonic motion,

$$\ddot{x} + \frac{k}{m} x = 0 \tag{1}$$

where x is the displacement from equilibrium. For our problem we define $t = 0$ as the moment that the brakes are applied to the cable. The solution to (1) is then

$$x = \frac{v_0}{\omega} \sin \omega t \tag{2}$$

where v_0 is the velocity of the mass at the instant of brake application, and $\omega = \sqrt{k/m}$. The peak force is, from $\ddot{x} = -\omega^2 x$,

$$F_{\max} = m \omega v_0 \tag{3}$$

1.1. Initial Velocity

Now we need to know two things: the velocity and the spring constant. From the equation of an object falling a short distance from rest, $\ddot{y} = g$, we have $\dot{y} = gt$ and $y = \frac{1}{2} g t^2$. Hence, we can write the velocity as $\dot{y} = \sqrt{2gy}$. The velocity after falling a distance L is then

$$v_0 = \sqrt{2gL} \tag{4}$$

1.2. Spring Constant — Bulk Modulus Approximation

For the spring constant, we (probably erroneously) consider the cable to be an isotropic medium with bulk modulus κ. The magnitude of the change in volume of the cable when subjected to a force F along its length is then

$$\Delta V = 2\pi R \Delta R + \pi r^2 \Delta L = \frac{F}{\pi R^2 \kappa} V = \frac{FL}{\kappa} \tag{5}$$

If we assume the change in radius is small compared to the change in length, we may write

$$\frac{\Delta L}{L} \approx \frac{F}{\pi R^2 \kappa} \tag{6}$$

Hence, the spring constant of the cable is

$$k = \frac{\pi R^2 \kappa}{L} \tag{7}$$

and we have

$$\omega = \sqrt{\frac{\pi R^2 \kappa}{mL}} \tag{8}$$

Using (4) and (8), eq. (2) becomes

$$x = \sqrt{\frac{2mg}{\pi R^2 \kappa}} L \sin \omega t \tag{9}$$

and the peak force is

$$F_{\max} = \sqrt{2\pi\, mg R^2 \kappa} \tag{10}$$

The maximum displacement from eq. (9), and hence the maximum stretching of the cable, is therefore

$$\Delta L = \sqrt{\frac{2mg}{\pi R^2 \kappa}} L \tag{11}$$

1.3. Modulus of Elasticity (Young's Modulus) Correction

One could take into account the ΔR neglected above by using Young's modulus E. We then have

$$\frac{\Delta L}{L} = \frac{F}{\pi R^2 E} \tag{12}$$

(compare to eq. (6)) giving us the spring constant

$$k = \frac{\pi R^2 E}{L} \tag{13}$$

and resulting maximum displacement

$$\boxed{\Delta L = \sqrt{\frac{2mg}{\pi R^2 E}} L} \tag{14}$$

and peak force

$$F_{\max} = \sqrt{2\pi\, mg R^2 E} \tag{15}$$

For steel, the bulk modulus is approximately

$$\kappa_{steel} = 16 - 17 \times 10^{10} \text{ Pa},$$

while the Young's modulus is

$$E_{steel} = 20 - 22 \times 10^{10} \text{ Pa}$$

Suppose $m = 30$ kg, $R = \frac{3}{32}$ in, and $L = 5$ m. Then, using eq. (14), we find that the cable would stretch about $6.1 - 6.4$ cm. The peak force would be in the range 10,400 to 10,900 lbs.

1.4. Peak Force as a Function of ΔL

For any linear spring, the peak force is proportional to the peak displacement by the spring constant,

$$\begin{aligned} F_{\max} &= k \cdot \Delta L \\ &= \pi R^2 E \frac{\Delta L}{L} \end{aligned} \tag{16}$$

But from (14),

$$\pi R^2 E = 2mg \left(\frac{L}{\Delta L} \right)^2 \tag{17}$$

So we have the simple relation

$$\boxed{F_{\max} = 2mg \frac{L}{\Delta L}} \tag{18}$$

which is conveniently independent of the cable diameter and modulus of elasticity.

2. MULTISTRAND CABLE

Now consider the effects of a multistrand cable. First, the weight gets distributed evenly among the N strands, $m \to \frac{m}{N}$, so eq. (1) becomes

$$\ddot{x} + \frac{k}{m} N x = 0 \tag{19}$$

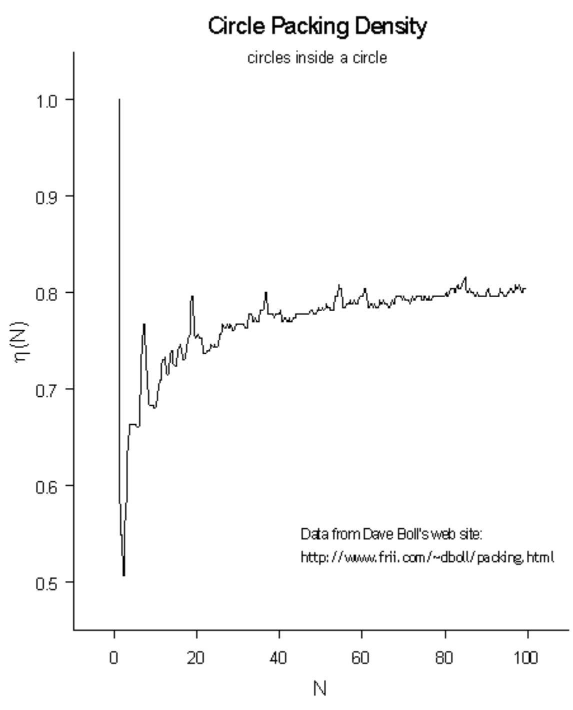
Each strand will then stretch an amount

$$\Delta L = \sqrt{\frac{2mg}{N\pi r^2 E}}\, L \tag{20}$$

where r is the individual strand radius.

Second, the radius R of the cable is now a function of 1) the number of strands, 2) the individual strand radius r, and 3) the packing density of the strands.

The packing density[1] $\eta(N) \le 1,\ N > 1,$ is the ratio of areas of the N strands ($N\pi r^2$) and the enclosing circle (πR^2). The packing density for our case is bounded: $\frac{1}{2} \le \eta(N) \le \frac{\pi\sqrt{3}}{6} \approx 0.9$. The lower bound is just $\eta(2)$. The upper bound is the densest possible packing of circles in the plane. The actual case of N circles of radius r filling a circle of radius R will always be between these bounds. Below is a plot[2] of the packing density of circles in a circle as a function of N.



For a reasonable number of cable strands, we have $\eta \approx \frac{3}{4}$.

We are now in a position to estimate the radius of an individual cable strand as a function of the number of strands and of the cable radius. Since

$$\eta(N) = \frac{N\pi r^2}{\pi R^2} \tag{21}$$

we may write

$$r = \sqrt{\frac{\eta(N)}{N}}\, R \tag{22}$$

Then eq. (20) becomes

$$\boxed{\Delta L = \sqrt{\frac{2mg}{\pi R^2 \eta E}}\, L} \tag{23}$$

Thus, the effect of multiple strands is, for a cable of fixed radius R, to weaken the "spring" by an amount proportional to $\eta^{-\frac{1}{2}}$.

In the case of our numerical example of the previous section (and using $\eta = \frac{3}{4}$), we find that the cable would stretch about $7.0 - 7.4$ cm. The peak force would be roughly 9050 to 9560 lbs.

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[1] See, e.g., http://www.astro.virginia.edu/~eww6n/math/CirclePacking.html (the online *CRC Concise Encyclopedia of Mathematics*), http://www.stetson.edu/~efriedma/packing.html, and http://www.maa.org/mathland/mathland_11_25.html

[2] Data derived from a graphic at Dave Boll's circle packing web page, http://www.frii.com/~dboll/packing.html