ON PREDICTING LONG-TERM ORBITAL INSTABILITY: A RELATION BETWEEN THE LYAPUNOV TIME AND SUDDEN ORBITAL TRANSITIONS

MYRON LECAR, FRED FRANKLIN, AND MARC MURISON

Harvard–Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138

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ABSTRACT

In three examples representative of Solar System dynamics, we find that the Lyapunov time, T_L (i.e., the inverse of the Lyapunov exponent) and the time for an orbit to make a sudden transition T_C are strongly correlated. The relation between the two times is $T_C \propto T_L^b$, with $b \cong 1.8$. The first example examines asteroid orbits interior to Jupiter; the sudden transition occurs when the asteroid makes a close approach to Jupiter, which occurs close to the time when the asteroid's orbit crosses Jupiter's orbit. The second example examines orbits of hypothetical asteroids between Jupiter and Saturn; the sudden transition occurs when the asteroid's orbit crosses the orbit of either of the planets. The third, considerably different, example examines massless bodies that initially orbit the smaller of the two masses of a binary system. In this case, the escape of the satellite signals the sudden transition. We have numerically integrated about 150 orbits in the first and second examples, and about 1000 orbits in the third example. In the first two examples, all three bodies were coplanar. In the third example, the initial inclination of the test particle was varied from 0° to 60°. There was, at most, a weak dependence on inclination. The tight clustering of the exponent b was remarkable, considering the widely different dynamical systems. The maximum departure of b from 1.8 was 14%, and the average departure was less than 7%. The correlation between T_C and T_L holds over at least six orders of magnitude in T_C; the longest integrations for asteroids interior to Jupiter extended for 10^8 yr. The Lyapunov time typically reaches its asymptotic value in a few thousand orbits, and then it can be used to predict sudden events (T_C) that occur at much later times, e.g., the lifetime of the solar system.

1. INTRODUCTION

We first noticed, in a study of the orbits of hypothetical asteroids between Jupiter and Saturn, that the Lyapunov time, T_L (the inverse of the Lyapunov exponent) and the time for the asteroid orbit to cross the orbit of Jupiter *or* Saturn T_C were correlated (Soper *et al.* 1990). Subsequently, we found, in studies of the orbits of hypothetical asteroids in the outer asteroid belt, that T_L was correlated with the time for the asteroid orbit to cross Jupiter's orbit (Lecar *et al.* 1991). In both cases, we found $T_C \propto T_L^b$, with $b \cong 1.7$, although the constant of proportionality was different in the two cases. We were finally convinced of the generality of the correlation by the 1000 orbits computed for this study by Murison. He considered elliptic restricted three-body orbits in which the massless particles begin motion around the secondary mass, which was 1/9 the mass of the primary and had an eccentricity of 0.1. In this example, the transition occurs when the particles escape the secondary by exiting through one of the colinear Lagrange points. Here too, $T_C \propto T_L^b$, and the average value of b was 1.8. This result held for massless particles in the plane of the binary, and inclined by 3°, 10°, 15°, 30°, 45°, and 60°. There may be a weak dependence on inclination.

The purpose of this paper is to present numerical evidence for the correlation. In all cases, the statistical significance of the correlation is strong. The weakest case had a linear correlation coefficient of 0.78 with 55 points. Put another way, the standard deviations of the exponent b were on average 6% of b, and in the worst case, 14% of b. Thus, in the worst case, $b=0$ is 9 standard deviations from the mean.

The validity of this experimental result does not rest at all on our tentative effort below to model the process. However, we would feel more secure if the numerical results were backed up by a quantitative theory. We are actively pursuing such a theory and we encourage our readers to do the same.

For the purposes of orientation, we proceed to outline a model. Consider orbits in the outer asteroid belt. If the orbits were unconstrained by isolating integrals, we would expect that the main effect of Jupiter would be to perturb the eccentricity of the asteroid until it crossed the orbit of Jupiter. A simple model that preserves the essential physics supposes that an asteroid, a distance d from Jupiter, receives a radial impulse at conjunction of magnitude

$$\Delta v \cong (GM_J/d^2)(2d/V_{\rm rel}),$$

where $V_{\rm rel} \cong Vd/2R$ and $V^2 = GM_\odot/R$. The change in the eccentricity is $\Delta e \cong \Delta v/V$, whence

$$\Delta e \cong 4\mu(R/d)^2, \quad \text{where } \mu = M_J/M_\odot.$$

If these changes occur with random sign, then, in the spirit of random walk calculations,

$$\frac{de^2}{dt} = \nu(\Delta e)^2,$$

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where $\nu = v_{\rm rel}/2\pi R = (d/2R)(1/T_{\rm J})$, and $T_{\rm J}$ is Jupiter's orbital period (11.8 yr). Thus, $e^2 = 8(\mu)^2(R/d)^3(t/T_{\rm J})$.

The asteroid's orbit will cross Jupiter's orbit when $e = (d/R)$. Thus,

$$t_{\rm C}/T_{\rm J} = (1/8\mu^2)(d/R)^5,$$

where $t_{\rm C}$ is the crossing time. For $\mu = \mu_{\rm Jup} = 1/1047$, and $d/R = 0.35$ (corresponding to asteroids with semimajor axes just outside of the 2:1 resonance with Jupiter), we have $t_{\rm C}/T_{\rm J} \cong 10^3$.

In contrast, the numerical integrations show that many orbits, with initially small eccentricities, do not become Jupiter crossers in 10^7 $T_{\rm J}$. This large discrepancy in time scales indicates that the numerically integrated orbits are constrained, for long periods of time, by almost-isolating integrals.

A picture we find intuitively appealing was offered by Froeschle & Scheidecker (1973). In a two-dimensional surface-of-section, they pictured a finite number of isolating integrals as nested ellipses, with an outermost ellipse surrounded by chaotic orbits. Perturbations blur the isolating integrals and it becomes possible for a particle to diffuse across the now-fuzzy nested ellipses. They indicate that the orbit random walks across the N ellipses, in N^2 steps. If their model is applicable to our numerical experiments, it implies that the Lyapunov time is a function of the distance of the orbit from the chaotic zone. This is certainly qualitatively correct; the Lyapunov time increases as the distance from the chaotic zone increases.

In the following sections, we treat the three examples separately; i.e., asteroids interior to Jupiter, asteroids between Jupiter and Saturn, and escaping satellites. The results of all the experiments are summarized in the concluding section.

2. ASTEROIDS INTERIOR TO JUPITER

We recently integrated a grid of asteroid orbits interior to Jupiter, with 14 semimajor axes ranging from 0.63 to 0.76 (in units where $a_{\rm Jupiter} = 1.0$) in increments of 0.01, and at 10 values of the eccentricity from 0.01 to 0.19 in increments of 0.02. The integrations extended for 1 million Jovian years (about 12 mil. yr), and in some cases for 10 mil Jovian yr, unless the asteroid crossed Jupiter's orbit first. Jupiter was in an elliptic orbit with its eccentricity varying in the 54 000 yr period, and with its line of apsides rotating with the 300 000 yr period induced by the secular perturbations of Saturn (Brouwer & Van Woerkom 1950). However, Saturn was not included in the integration. All bodies were confined to a plane.

In following the progress of integrations on the computer screen, we noticed that a crossing of Jupiter's orbit was shortly followed by a close approach to Jupiter. We documented the difference between the time it took to cross Jupiter's orbit and a very close approach to within 5 Jovian radii of Jupiter for a small sample of eight cases: it was 2% on the average and always less than 6% (Franklin *et al.* 1989). Details of the method of numerical integration can also be found in that paper. For computational convenience, we may therefore use the orbit crossing criterion, except in cases of resonance, where an asteroid's orbit can cross Jupiter's orbit and still avoid a close approach. In that case, we use the criterion of close approach. Thus, $T_{\rm C}$ is defined as the time when the asteroid's orbit crosses Jupiter's orbit, $R > 1.0$, except in the cases of resonance.

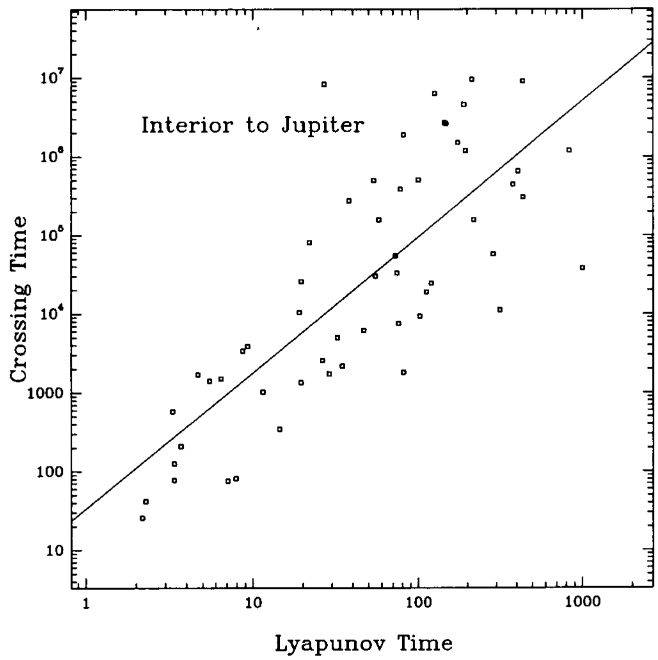


FIG. 1. $\log(T_{\rm C})$ vs $\log(T_{\rm L})$ for 55 orbits of test particles in the outer asteroid belt. A least-squares fit of a straight line to these data, $\log(T_{\rm C}) = a + b\log(T_{\rm L})$, yields $b = 1.73$, $\sigma_b = 0.19$, $a = 1.53$, and $\sigma_a = 0.34$. The longest integrations extend for 10^7 Jovian yr (about 1.2×10^8 yr).

To calculate the Lyapunov exponent, we computed the difference in longitude $\delta\lambda$ between two orbits that started with the same initial conditions except for an initial longitude difference, $\delta\lambda_0 = 10^{-6}$ deg. For unstable orbits, the subsequent behavior of $\delta\lambda$ is well represented by $\delta\lambda(t) = \delta\lambda_0 \exp(\gamma t) = \delta\lambda_0 \exp(t/T_{\rm L})$. We were particularly interested in whether or not the Lyapunov exponent calculated at the start of the integration could be used to predict the crossing time. For that reason, $T_{\rm L}$ was computed at the start of the integration, using the first 100 000 Jupiter periods (less if saturation occurred sooner). The portion of the integration that was used in the calculation of $T_{\rm L}$ was always much shorter than $T_{\rm C}$.

Figure 1 displays a plot of $\log(T_{\rm C})$ vs $\log(T_{\rm L})$ for 55 points from this study (time units are Jovian years). A least-squares fit of a straight line to these data,

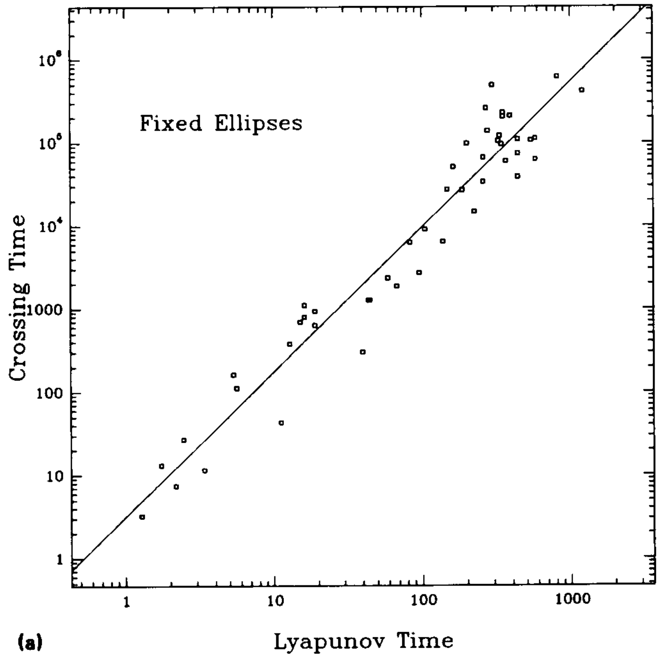
$$\log(T_{\rm C}) = a + b\log(T_{\rm L})$$

yields $b = 1.73$, $\sigma_b = 0.19$, $a = 1.53$, and $\sigma_a = 0.34$.

Note that the numerical value of the intercept a depends on the unit of time. Explicitly, we have

$$\log(T_{\rm C}/T_0) = a + b\log(T_{\rm L}/T_0),$$

where T_0 is the unit of time. We discuss, in the section on the third example, why $T_{\rm Jup}$ is the appropriate unit of time.



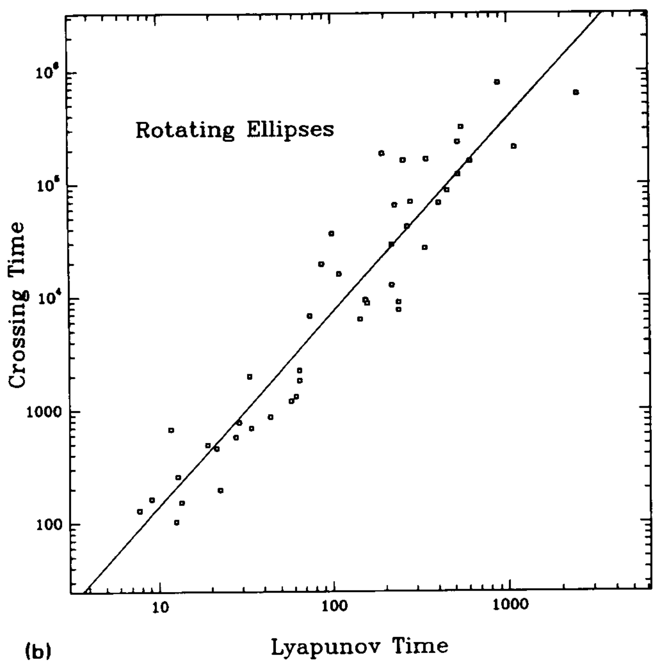


FIG. 2. (a) $\log(T_C)$ vs $\log(T_L)$ for 48 test particle orbits between Jupiter and Saturn, with the planets in fixed ellipses. The longest crossing times were 10^6 Jovian yr. In this example, $b=1.75$, $\sigma_b=0.07$, $a=0.50$, $\sigma_a=0.14$. Within the errors, the exponent b is equal to that of the previous example, but the intercept a is not. (b) $\log(T_C)$ vs $\log(T_L)$ for 45 test particles between Jupiter and Saturn, with the planets in the rotating ellipses with varying eccentricities. In this example, $b=1.72$, $\sigma_b=0.09$, $a=0.42$, and $\sigma_a=0.19$.

Scatter in $\log(T_C)$ is intrinsic to the problem. The orbits that cross Jupiter's orbit are chaotic. Small changes in the initial conditions, in the integration routine, or in the perturbations result in orbits that depart from each other exponentially. Thus, the crossing time is a statistical quantity with intrinsic error bars.

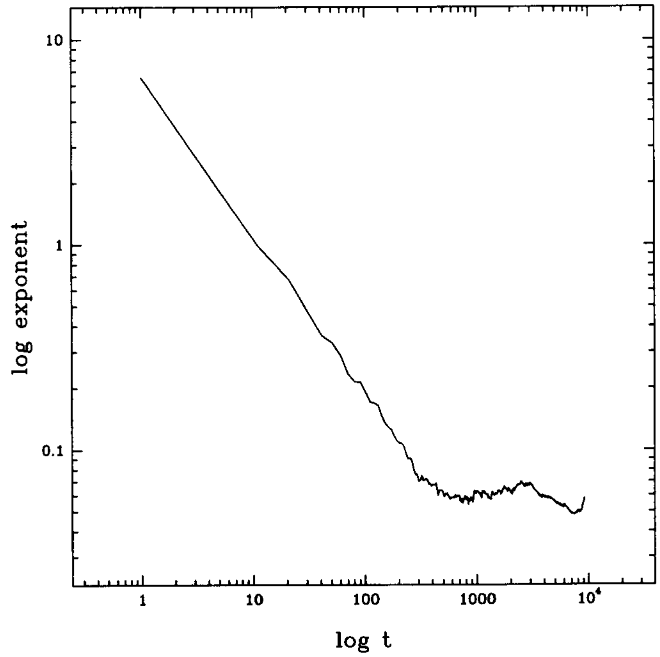


FIG. 3. Typical behavior of the Lyapunov exponent with time. Time is in units of the binary period.

3. ASTEROIDS BETWEEN JUPITER AND SATURN

We did this problem twice: once with Jupiter and Saturn in nonrotating, constant eccentricity ellipses, and once with the eccentricities varying and the line of apsides rotating according to the secular perturbations given by Brouwer and van Woerkem. Specifically,

$$h=e\sin(\omega)=A\sin(\alpha t)+B\sin(\beta t)$$

and

$$k=e\cos(\omega)=A\cos(\alpha t)+B\cos(\beta t),$$

where $A_J=0.0445$, $B_J=-0.0165$, $A_S=0.0360$, and $B_S=0.0490$. The periods $2\pi/\alpha=306\,000$ yr (25 780 T_J) and $2\pi/\beta=46\,000$ yr (3875 T_J).

Figure 2(a) displays $\log(T_C)$ vs $\log(T_L)$ for the fixed ellipse case. In this example, $b=1.75$, $\sigma_b=0.07$, $a=0.50$, and $\sigma_a=0.14$. Again, we note that the numerical value of the intercept, a, depends on the unit of time, which in this case is T_{Jup}.

Figure 2(b) displays $\log(T_C)$ vs $\log(T_L)$ for the rotating ellipse case. The 45 points are correlated in the same way with $b=1.72$, $\sigma_b=0.09$, $a=0.42$, and $\sigma_a=0.19$.

4. ESCAPE OF SATELLITES OF THE SMALLER MASS OF A BINARY SYSTEM

Murison (1989b) has previously investigated escape orbits in the circular restricted problem. The method of integration is described in Murison (1989a). For this study, $\mu=M_2/M=0.1$ where $M=M_1+M_2$, and the unit of time is the binary period. The secondary M_2 was placed in an elliptic orbit with $e=0.1$ in most of the cases, but in one case we took $e=0.6$. The massless particle started on the

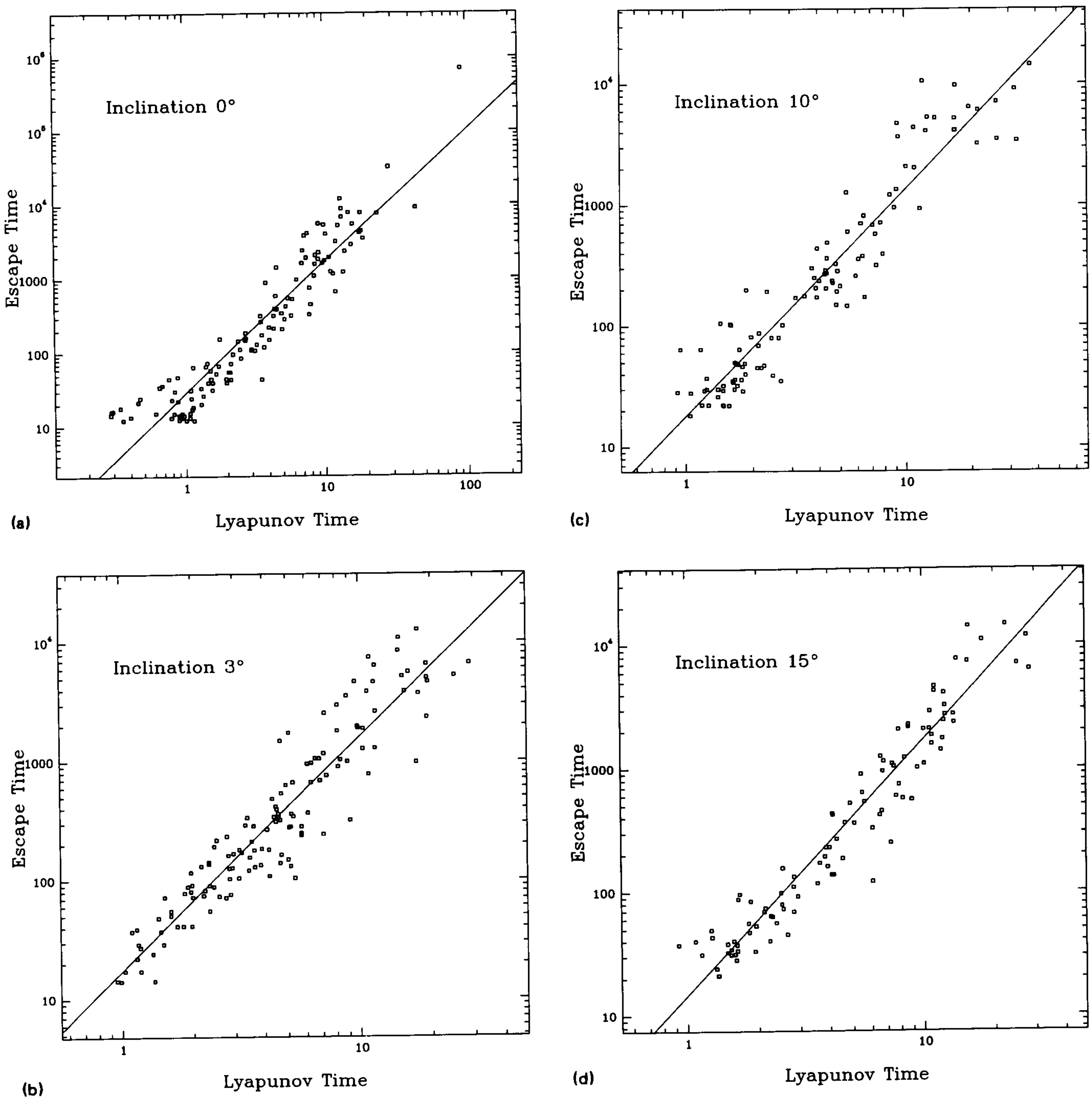
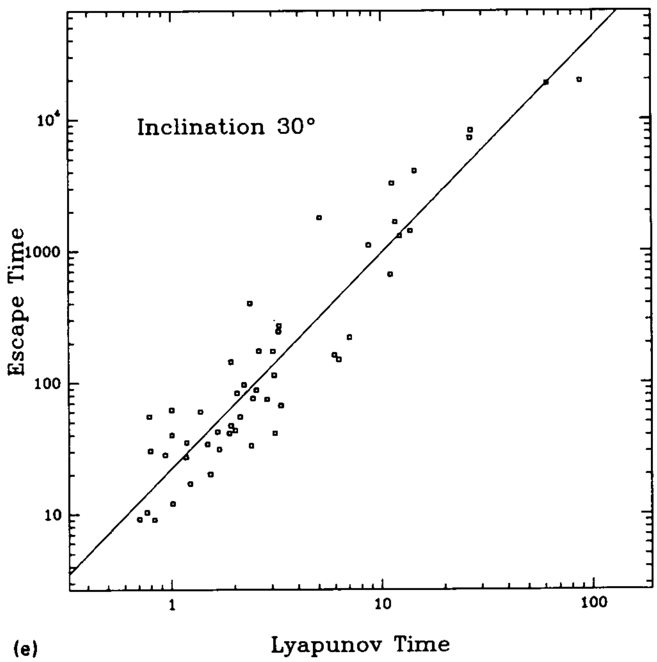
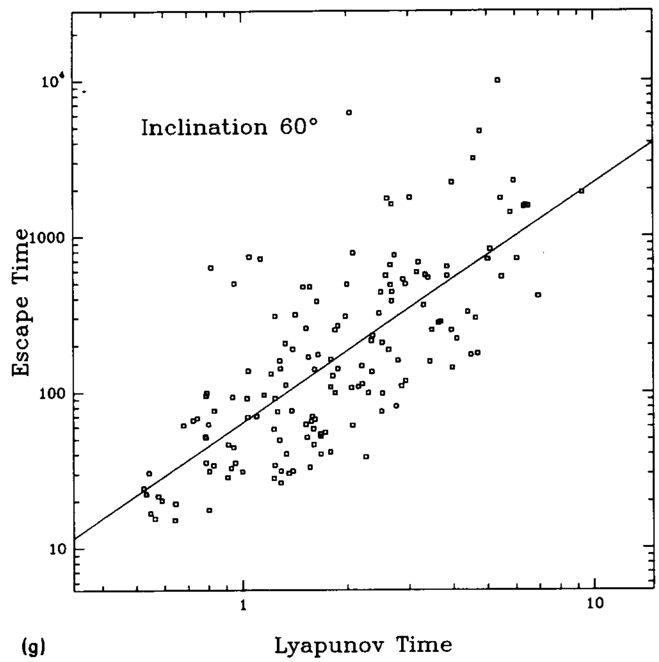


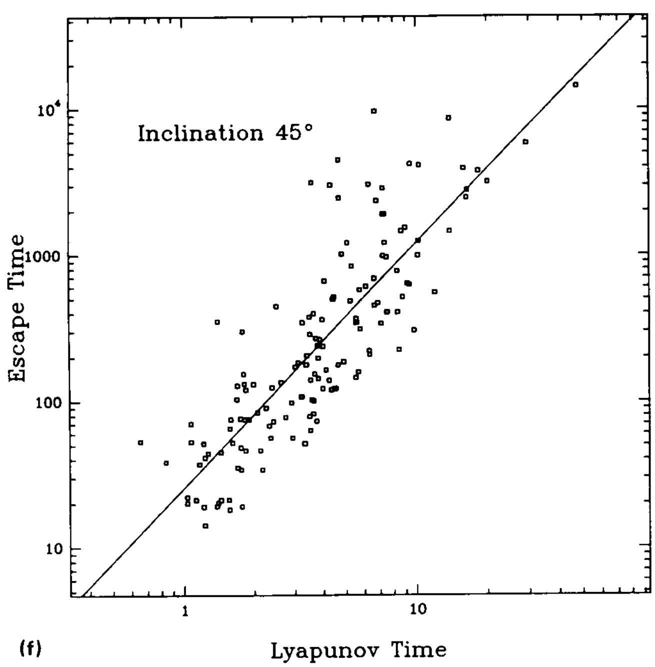
FIG. 4. (a)–(g) $\log(T_C)$ vs $\log(T_L)$ results for the escape of satellites from the secondary of a binary system. Time is in units of the binary period. (a) is the coplanar case, while (b)–(g) are 3°, 10°, 15°, 30°, 45°, and 60° inclination, respectively.



(e)



(g)



(f)

FIG. 4. (continued)

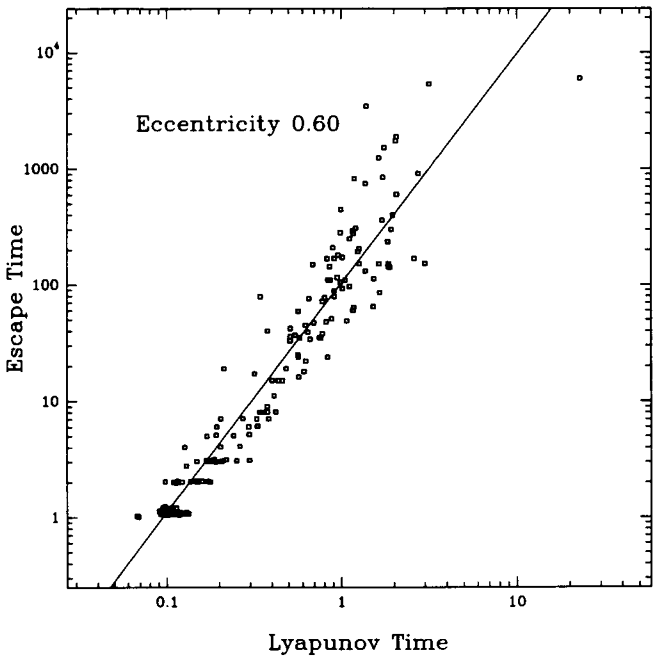


FIG. 5. $\log(T_C)$ vs $\log(T_L)$ for the escape of satellites from a secondary in a highly elliptic orbit with $e=0.6$. The orbits were coplanar.

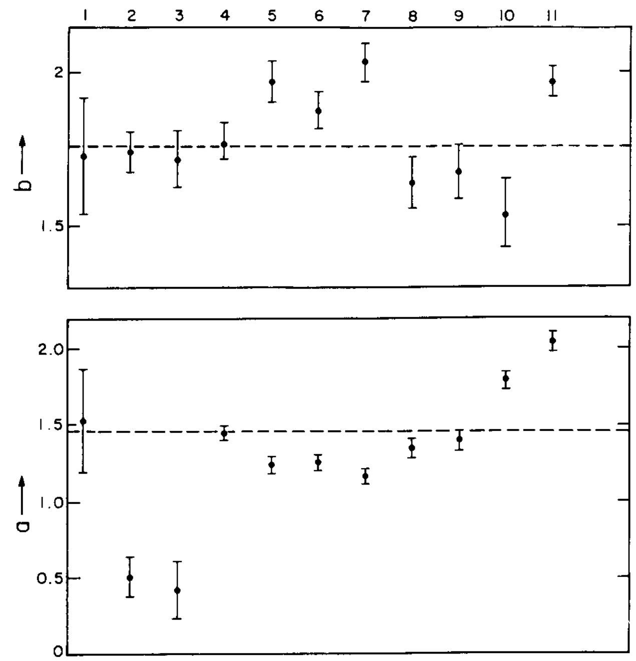


FIG. 6. Values and standard errors of the slope b and the intercept a for the 11 examples. The average value of b was 1.79. In all cases, $\log(T_C)=a+b\log(T_L)$ is a very good fit to the data. The slopes have a value close to 1.8. However, the intercept for the Jupiter–Saturn case is different than the intercept for the other cases.

line connecting the primary and secondary (the x axis) and on the side of the secondary away from the primary. The initial distance from the secondary, a_0, was 0.125 in units where binary semimajor axis is one. The initial velocity was along the positive y axis (prograde motion). The initial eccentricity of the test particle varied from 0.25 to 0.45. Initial eccentricities less than 0.25 had very long escape times ($T_C > 10^6\ T_{\text{binary}}$). The inner and outer Lagrange points are at 0.709 and 1.360. We note that even when the secondary is at periastron, the distance from the instantaneous inner Lagrange point to the secondary is

TABLE 1. Summary of statistics: Least-square fit of the relation $\log(T_C)=a+b\log(T_L)$, N=number of orbits, b=exponent, σ_b=standard error of b, a=intercept, σ_a=standard error of a, r=linear correlation coefficient.

Description		N	b	σ_b	a	σ_a	r
Interior to Jupiter		55	1.73	0.19	1.53	0.34	0.78
Jupiter and Saturn; fixed ellipses		48	1.75	0.07	0.50	0.14	0.91
Jupiter and Saturn; rotating ellipses		45	1.72	0.09	0.42	0.19	0.95
Satellites:							
$e=0.1$	$i=0°$	126	1.78	0.06	1.45	0.04	0.94
	$i=3°$	130	1.97	0.07	1.24	0.05	0.94
	$i=10°$	107	1.88	0.06	1.25	0.05	0.95
	$i=15°$	100	2.04	0.06	1.17	0.05	0.96
	$i=30°$	51	1.64	0.09	1.35	0.06	0.94
	$i=45°$	144	1.68	0.09	1.41	0.06	0.84
	$i=60°$	154	1.54	0.12	1.80	0.05	0.74
$e=0.6$	$i=0°$	163	1.97	0.05	2.02	0.03	0.95
Average value			1.79	0.09	1.47[a]	0.08[a]	0.90

[a]Without the two Jupiter and Saturn cases.

0.262 which is more than twice a_0. Thus, to cross a Lagrange point (i.e., to escape) the test particle must increase its semimajor axis as well as its eccentricity.

Integrations were terminated when the distance of the test particle from the secondary was $\geqslant 0.5$. The largest Lyapunov exponent was calculated in the usual way using the distance in phase space and averaging over the entire integration. Periodic rescaling removed the problem of saturation (cf. Benettin *et al.* 1976). In long integrations of order 10^6 orbits, the Lyapunov exponent tended to an asymptotic value in about 10^3 orbits, which explains why these data are consistent with the earlier examples. In shorter integrations, the Lyapunov exponent reached its aymptotic value before escape occured. Figure 3 shows typical behavior of the Lyapunov exponent with time.

Figure 4(a) displays the $\log(T_C)$ vs $\log(T_L)$ results for the coplanar case. The same relation holds between $\log(T_C)$ and $\log(T_L)$ with $b=1.78$, $\sigma_b=0.06$, $a=1.45$, and $\sigma_a=0.04$.

Here, the unit of time is the binary period, and we notice that the numerical value of the intercept is consistent with the value found in the first example. The dynamical system in both examples consists of a dominant central mass with one significant perturber.

We were curious to see if the exponent b depended on the number of degrees of freedom. We therefore integrated sets of orbits with inclinations of 3°, 10°, 15°, 30°, 45°, and 60°. Figures 4(b)–4(f) display $\log(T_C)$ vs $\log(T_L)$ for these cases. It appears there may be a weak dependence on inclination, though the errors are consistent with no inclination dependence.

Finally, we integrated an extreme coplanar case where the eccentricity of the secondary was 0.6. $\mathrm{Log}(T_C)$ vs $\log(T_L)$ for 163 integrations is presented in Fig. 5. This case was also consistent with the previous case.

Table 1 summarizes the statistics for all the cases.

5. CONCLUSIONS

The striking feature of these results is the tight clustering of the values of b near 1.8 in the relation $T_C \propto T_L^b$. Figure 6 displays the values and standard errors of b and a. To apply this result to other problems, one must remember that the unit of time determines the numerical value of the constant of proportionality. We adopt the following prescription:

$$(T_C/T_0) = A(T_L/T_0)^{1.8},$$

where, in the case of two massive bodies, the natural choice for T_0 is the binary period.

This relation holds over a range of *six* orders of magnitude in T_C, and in a variety of dynamical problems. This helps to explain the somewhat puzzling results of Sussman and Wisdom (1988), who found a Lyapunov time for Pluto of only 20 Myr in an integration of 800 Myr, and in which Pluto did not make a close approach to another planet. In this case, we take the period of Neptune (165 yr) as the unit of time, set $A=30$, and predict a planet crossing in 7×10^{12} yr. Also, Laskar (1989) found that the Lyapunov time for the inner solar system was only 5 Myr, yet we believe the planets in the inner solar system avoid close approaches for much longer times. In this case, we take the year as the unit of time and set $A=3$; the time for a planet crossing is then predicted to be 3×10^{12} years. Even allowing for some ambiguity in the choice of the time unit and therefore the value of A, these predictions provide a comfortable margin.

Finally, we ask if this relation applies to orbits in strong resonances. We have made some initial explorations of this problem. For nine orbits at the 3/2 resonance with Jupiter, the relation held quite well, with an average value of $A=37$. The relation also held at the 5/3, 8/5, and 7/4 resonances. On the other hand, we found no Jupiter crossers at the 2/1 resonance in 10^6 Jovian periods. For the 3/1 resonance, we turn to Wisdom's study (1983), where he found a typical Lyapunov time of 270 Jovian periods for orbits which had a spike in their eccentricity in about 20 000 Jovian periods. If our relation holds with the same exponent, $b=1.8$, then $A=1$ at the 3/1 resonance. Thus, the predictive power of the relation at strong resonances is problematic and deserves further study.

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