

Pseudo-High-Order Symplectic Integrators

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ABSTRACT

Symplectic N-body integrators are widely used to study problems in celestial mechanics. The most popular algorithms are of 2nd and 4th order, requiring 2 and 6 substeps per timestep, respectively. The number of substeps increases rapidly with order in timestep, rendering higher-order methods impractical. However, symplectic integrators are often applied to systems in which perturbations between bodies are a small factor ϵ of the force due to a dominant central mass. In this case, it is possible to create optimized symplectic algorithms that require fewer substeps per timestep. This is achieved by only considering error terms of order ϵ , and neglecting those of order ϵ^2 , ϵ^3 etc. Here we devise symplectic algorithms with 4 and 6 substeps per step which effectively behave as 4th and 6th-order integrators when ϵ is small. These algorithms are more efficient than the usual 2nd and 4th-order methods when applied to planetary systems.

Subject headings: celestial mechanics, stellar dynamics—methods: n-body simulations—methods: numerical

1. Introduction

Symplectic integrators are widely used to study problems in celestial mechanics. These integrators have two advantages over most other algorithms. First, they exhibit no long-term build up in energy error. Second, the motion of each object about the central body can be “built in”, so that the choice of step size, τ , is determined by the perturbations between bodies, whose magnitude is a factor ϵ smaller than the forces due to the central body ((Wisdom and Holman 1991)).

The most popular algorithm is the second-order symplectic integrator. The error at each step is proportional to $\epsilon\tau^3$, so that the likely error for an integration as a whole is $\sim \epsilon\tau^2$. The second-order method is easy to implement, consisting of only two substeps, including one force evaluation, per time step. It is also very fast for integrations requiring moderate accuracy.

For more accurate integrations, it is better to use the fourth-order method ((Forest and Ruth 1990)). Here, the error at each step is proportional to $\epsilon\tau^5$, although each step is computationally more expensive since it consists of 6 substeps. Yoshida (1990) has developed 6th and 8th-order symplectic integrators. However, these do not appear to be competitive in most situations, due to the large number of substeps required.

Here we show how to construct what are effectively high-order (4th, 6th etc.) symplectic integrators that require fewer substeps per time step than those in current use. The trick is to take into account the dependence of each error term on ϵ when choosing the coefficients for each substep. The algorithms are designed by eliminating error terms proportional to ϵ up to the desired order of the timestep. Error terms proportional to ϵ^2, ϵ^3 etc., in low orders of the timestep, still exist. However, in many situations these terms are negligible, and the integrators behave as if they are of higher order than the leading error term in τ suggests.

Section 2 gives a quick review of how symplectic integrators are traditionally constructed using Lie algebra. In Section 3, we show how to build more efficient symplectic algorithms using fewer substeps. Section 4 contains results of test integrations that compare the new algorithms with traditional symplectic integrators.

2. Symplectic Integrators

Symplectic integrators for the N-body problem can be constructed starting from Hamilton's equations of motion:

$$\begin{aligned}\frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i}\end{aligned}\tag{1}$$

where x_i and p_i are the coordinates and momenta of each body respectively, and H is the Hamiltonian of the system.

Using these equations, the rate of change of any dynamical quantity $q(\mathbf{x}, \mathbf{p}, t)$ can be expressed as

$$\frac{dq}{dt} = \sum_{i=1}^{3N} \left(\frac{\partial q}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial q}{\partial p_i} \frac{\partial H}{\partial x_i} \right) \equiv \{q, H\} \equiv Fq\tag{2}$$

where $\{, \}$ are Poisson brackets, and F is a differential operator.

The formal solution of equation (2) is

$$q(t) = e^{\tau F} q(t - \tau) = \left(1 + \tau F + \frac{\tau^2 F^2}{2} + \dots \right) q(t - \tau)$$

where $F^2 q = F(Fq)$ etc.

Now suppose that we are able to split the Hamiltonian into two pieces, H_A and H_B , so that each part of the problem can be solved relatively easily in the absence of the other. The solution for q becomes

$$q(t) = e^{\tau(A+B)}q(t - \tau) \quad (3)$$

where A and B are differential operators related to H_A and H_B respectively, in the same way that F is related to H .

The Baker-Campbell-Hausdorff (BCH) formula states that, for any noncommutative operators A and B ,

$$\exp(A) \cdot \exp(B) = \exp(C)$$

where C is a series consisting of nested commutators,

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, A, B] + \frac{1}{12}[B, B, A] + \dots$$

where the commutator $[A, B] = AB - BA \neq 0$ in general (see, for example, Yoshida 1990 or Forest and Ruth 1990). Here, we have used the nested commutator notation $[A, B, C] = [A, [B, C]]$, etc..

Hence, if we evolve q under the two parts of the Hamiltonian separately, one after the other, we have

$$\exp(\tau A) \cdot \exp(\tau B)q(t - \tau) = \exp\left[\tau F + \frac{\tau^2}{2}[A, B] + \dots\right]q(t - \tau) \quad (4)$$

This is identical to the righthand side of equation (3) to $O(\tau)$, and so equation (4) represents a first-order integrator. Each step of the integrator consists of 2 substeps, with the whole step giving an error of $O(\tau^2)$. Alternatively, we can say that the integrator exactly solves a problem whose Hamiltonian is given by

$$H_{integ} = H + \frac{\tau}{2}\{H_B, H_A\} + O(\tau^2)$$

(see, for example, (Saha and Tremaine 1992)). Provided that τ is small, and $\{H_B, H_A\}$ remains bounded, the energy of the integrated system will always be near to that of the real system.

Other integrators can be found by combining exponential operators in such a way that they are equivalent to equation (3) up to a given order in τ . For example, we have the second-order symplectic integrator

$$\begin{aligned} \text{S2A} &= \exp\left(\frac{\tau}{2}A\right) \cdot \exp(\tau B) \cdot \exp\left(\frac{\tau}{2}A\right) \\ &= \exp\left[\tau F + \frac{\tau^3}{12}[B, B, A] - \frac{\tau^3}{24}[A, A, B] + \dots\right] \end{aligned}$$

When many integration steps are performed one after another, the $\exp(\tau A/2)$ terms at the end of one step and the start of another can be combined. Hence, the second-order integrator also consists of only 2 substeps, except at the beginning and the end of an integration.

Another second-order integrator is

$$\begin{aligned} \text{S2B} &= \exp\left(\frac{\tau}{2}B\right) \cdot \exp(\tau A) \cdot \exp\left(\frac{\tau}{2}B\right) \\ &= \exp\left[\tau F + \frac{\tau^3}{12}[A, A, B] - \frac{\tau^3}{24}[B, B, A] + \dots\right] \end{aligned} \quad (5)$$

The distinction between S2A and S2B (which at first sight appear to be the same) will become apparent in the next section, when we consider situations in which A and B are of different magnitude.

Forest and Ruth (1990) give a fourth-order symplectic integrator with 6 substeps per step:

$$\begin{aligned} \text{S4B} &= \exp\left(\frac{\tau B}{2c}\right) \cdot \exp\left(\frac{\tau A}{c}\right) \cdot \exp\left[\frac{\tau B(1-k)}{2c}\right] \cdot \exp\left(\frac{-\tau k A}{c}\right) \cdot \exp\left[\frac{\tau B(1-k)}{2c}\right] \\ &\cdot \exp\left(\frac{\tau A}{c}\right) \cdot \exp\left(\frac{\tau B}{2c}\right) \\ &= \exp[\tau F + O(\tau^5)] \end{aligned}$$

where $k = 2^{1/3}$ and $c = 2 - k$. Note that the middle 3 substeps move in the opposite direction to the integration as a whole.

Higher-order integrators require progressively more substeps. Yoshida (1990) gives examples of 6th and 8th-order integrators using 14 and 30 substeps respectively. In the next section, we will show how to create what are *effectively* 4th and 6th order integrators (and in principle, 8th-order etc.) using fewer substeps than are required conventionally.

3. Constructing Pseudo-Order Integrators

Up to this point we have not considered the details of how H is split. Suppose that one part of the Hamiltonian is much smaller than the other, *i.e.* $H = H_A + \epsilon H_B$, where $\epsilon \ll 1$. Now consider the error terms in the second-order integrator of equation (5):

$$\text{S2B} = \exp\left[\tau F + \frac{\epsilon\tau^3}{12}[A, A, B] - \frac{\epsilon^2\tau^3}{24}[B, B, A] + \dots\right]$$

One of the $O(\tau^3)$ error terms is smaller than the other by a factor of ϵ .

Similarly, for the fourth order integrator:

$$\text{S4B} = \exp[\tau F + O(\epsilon\tau^5) + O(\epsilon^2\tau^5) + O(\epsilon^3\tau^5) + O(\epsilon^4\tau^5)]$$

Some of these error terms are insignificant compared to others, and yet this was not taken into account when constructing the integrator. The only design criterion was that S4B should contain no error terms below the fifth power in the timestep. If we take into account the dependence of the error terms on both τ and ϵ , we can design more efficient symplectic integrators.

To construct the new integrators, we again employ the BCH formula. Adapting the expression for the BCH formula given by Yoshida (1990), we have:

$$\begin{aligned} & \exp(a_1\tau A) \cdot \exp(b_1\epsilon\tau B) \\ = & \exp \left[(a_1A + \epsilon b_1B)\tau + \epsilon\tau^2 \left(\frac{a_1b_1}{2} \right) [A, B] + \epsilon\tau^3 \left(\frac{a_1^2b_1}{12} \right) [A, A, B] + \epsilon^2\tau^3 \left(\frac{a_1b_1^2}{12} \right) [B, B, A] \right. \\ & \left. + \epsilon^2\tau^4 \left(\frac{a_1^2b_1^2}{24} \right) [A, B, B, A] - \epsilon\tau^5 \left(\frac{a_1^4b_1}{720} \right) [A, A, A, A, B] - \epsilon^4\tau^5 \left(\frac{a_1b_1^4}{720} \right) [B, B, B, B, A] + \dots \right] \end{aligned}$$

where a_1 and b_1 are constants. Additional fifth-order commutators are present; however, we will only require terms that contain either A or B once, since these are the type of error term we are seeking to eliminate.

Applying the BCH formula twice, Yoshida (1990) gives an expression for a symmetric product of three exponential operators:

$$\begin{aligned} & \exp(b_1\epsilon\tau B) \cdot \exp(a_1\tau A) \cdot \exp(b_1\epsilon\tau B) \\ = & \exp \left[(a_1A + 2\epsilon b_1B)\tau + \epsilon\tau^3 \left(\frac{a_1^2b_1}{6} \right) [A, A, B] - \epsilon^2\tau^3 \left(\frac{a_1b_1^2}{6} \right) [B, B, A] \right. \\ & \left. - \epsilon\tau^5 \left(\frac{a_1^4b_1}{360} \right) [A, A, A, A, B] + \epsilon^4\tau^5 \left(\frac{7a_1b_1^4}{360} \right) [B, B, B, B, A] + \dots \right] \end{aligned} \quad (6)$$

Again we have neglected fifth-order terms that contain both A and B more than once. Note that there are no terms containing even powers of the timestep: Yoshida shows that this is a general property of any symmetric arrangement of exponential operators. From now on we will consider only symmetrical integrators because of this property.

We need to extend equation (6) once more to get a pseudo-fourth order integrator, and twice more for a pseudo-sixth order one. By substituting a_2A for b_1B in equation (6), and substituting the righthand side of equation (6) for a_1A , we get:

$$\begin{aligned} & \exp(a_2\tau A) \cdot \exp(b_1\epsilon\tau B) \cdot \exp(a_1\tau A) \cdot \exp(b_1\epsilon\tau B) \cdot \exp(a_2\tau A) \\ = & \exp \left\{ (a_1 + 2a_2)\tau A + 2b_1\epsilon\tau B + \epsilon\tau^3 \left(\frac{b_1}{6} \right) [(a_1 + 2a_2)^2 - 6a_2(a_1 + a_2)] [A, A, B] \right. \\ & + \epsilon^2\tau^3 \left(\frac{b_1^2}{6} \right) (4a_2 - a_1) [B, B, A] - \epsilon\tau^5 \left(\frac{b_1}{360} \right) [(a_1 + 2a_2)^4 - 30a_2^2(a_1 + a_2)^2] [A, A, A, A, B] \\ & \left. - \epsilon^4\tau^5 \left(\frac{b_1^4}{360} \right) (16a_2 - 7a_1) [B, B, B, B, A] + \dots \right\} \end{aligned} \quad (7)$$

Finally, substituting the righthand side of equation (7) for a_1A in equation (6), and replacing b_1B with b_2B , we arrive at

$$\exp(b_2\epsilon\tau B) \cdot \exp(a_2\tau A) \cdot \exp(b_1\epsilon\tau B) \cdot \exp(a_1\tau A) \cdot \exp(b_1\epsilon\tau B) \cdot \exp(a_2\tau A) \cdot \exp(b_2\epsilon\tau B)$$

$$\begin{aligned}
&= \exp \left\{ (a_1 + 2a_2)\tau A + 2(b_1 + b_2)\epsilon\tau B + \epsilon\tau^3 \left[\frac{(b_1 + b_2)(a_1 + 2a_2)^2 - 6a_2b_1(a_1 + a_2)}{6} \right] [A, A, B] \right. \\
&- \epsilon^2\tau^3 \left[\frac{(a_1 + 2a_2)(b_1 + b_2)^2 - 6a_2b_1^2}{6} \right] [B, B, A] \\
&- \epsilon\tau^5 \left[\frac{(b_1 + b_2)(a_1 + 2a_2)^4 - 30a_2^2b_1(a_1 + a_2)^2}{360} \right] [A, A, A, A, B] \\
&+ \left. \epsilon^4\tau^5 \left[\frac{7(a_1 + 2a_2)(b_1 + b_2)^4 - 60a_2b_1^2(b_1 + b_2)^2 + 30a_2b_1^4}{360} \right] [B, B, B, B, A] + \dots \right\} \quad (8)
\end{aligned}$$

The first stage in converting these general expressions into specific integrators is to make the coefficients of the linear A and B terms equal to 1. This places two constraints on the values of the constants. We can then get what is effectively a 4th-order integrator by simply eliminating the $[A, A, B]$ term from equation (7). The leading error terms will now be $O(\epsilon^2\tau^3)$ and $O(\epsilon\tau^5)$. Provided that ϵ is small enough, the largest error term will be $O(\epsilon\tau^5)$, and the integrator effectively will be of fourth order in the timestep. Applying these conditions, we require

$$\begin{aligned}
a_1 + 2a_2 &= 1 \\
2b_1 &= 1 \\
1 - 6a_2(1 - a_2) &= 0
\end{aligned} \quad (9)$$

where we have used the first two of equations (9) in deriving the third.

Alternatively, we may construct an integrator in which each step begins by advancing H_B instead of H_A . Unlike conventional symplectic integrators, such as $S2A$ and $S2B$, we cannot use the same set of coefficients when exchanging A and B . Instead, we must derive a new set of coefficients by interchanging A and ϵB in equation (7) and then eliminating the new $[A, A, B]$ term. When we do this, the first two of equations (9) remain as before, but the third expression becomes

$$6a_2 - 1 = 0 \quad (10)$$

To get a pseudo-6th-order integrator, we eliminate terms containing $[A, A, B]$ and $[A, A, A, A, B]$. This will produce an extra constraining equation, so we need an extra constant. We get this by using an integrator with the form of equation (8) instead of equation (7). The corresponding equations for the constants are

$$\begin{aligned}
a_1 + 2a_2 &= 1 \\
2(b_1 + b_2) &= 1 \\
1/2 - 6a_2b_1(1 - a_2) &= 0 \\
1/2 - 30a_2^2b_1(1 - a_2)^2 &= 0
\end{aligned} \quad (11)$$

If we prefer an integration step that begins by advancing H_A , we can interchange A and ϵB in equation (8), and eliminate the new $[A, A, B]$ and $[A, A, A, A, B]$ terms. In this case, the last two

of equations (11) become

$$\begin{aligned} 1/4 - 6a_2b_1^2 &= 0 \\ 7/16 - 15a_2b_1^2 + 30a_2b_1^4 &= 0 \end{aligned} \tag{12}$$

The leading error terms for each of these integrators are $O(\epsilon^2\tau^3)$ and $O(\epsilon\tau^7)$. The latter will be dominant if ϵ is small enough, so that the algorithms behave as 6th-order integrators.

3.1. Pseudo-4th and 6th-Order Examples

Solving equations (9) and (10), we obtain two pseudo-4th-order integrators:

$$\begin{aligned} \text{S4A*} &= \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right] \cdot \exp\left(\frac{\epsilon\tau B}{2}\right) \cdot \exp\left(\frac{\tau A}{\sqrt{3}}\right) \cdot \exp\left(\frac{\epsilon\tau B}{2}\right) \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right] \\ &= \exp\left[\tau F + \epsilon^2\tau^3\left(\frac{2-\sqrt{3}}{24}\right)[B, B, A] - \frac{\epsilon\tau^5}{4320}[A, A, A, A, B] + \dots\right] \\ \text{S4B*} &= \exp\left(\frac{\epsilon\tau B}{6}\right) \cdot \exp\left(\frac{\tau A}{2}\right) \cdot \exp\left(\frac{2\epsilon\tau B}{3}\right) \cdot \exp\left(\frac{\tau A}{2}\right) \cdot \exp\left(\frac{\epsilon\tau B}{6}\right) \\ &= \exp\left[\tau F + \frac{\epsilon^2\tau^3}{72}[B, B, A] + \frac{\epsilon\tau^5}{2880}[A, A, A, A, B] + \dots\right] \end{aligned}$$

where the asterisk in S4A* indicates that it only behaves as a 4th order integrator for certain values of τ .

Equations 11 and 12 give two pseudo-6th-order integrators:

$$\begin{aligned} \text{S6A*} &= \exp\left[\frac{\tau A}{2}\left(1 - \frac{3}{\sqrt{15}}\right)\right] \cdot \exp\left(\frac{5\epsilon\tau B}{18}\right) \cdot \exp\left(\frac{3\tau A}{2\sqrt{15}}\right) \cdot \exp\left(\frac{4\epsilon\tau B}{9}\right) \cdot \exp\left(\frac{3\tau A}{2\sqrt{15}}\right) \\ &\quad \cdot \exp\left(\frac{5\epsilon\tau B}{18}\right) \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{3}{\sqrt{15}}\right)\right] \\ &= \exp\left[\tau F + \epsilon^2\tau^3\left(\frac{54-13\sqrt{15}}{648}\right)[B, B, A] + O(\epsilon\tau^7)\right] \\ \text{S6B*} &= \exp\left(\frac{\epsilon\tau B}{12}\right) \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right] \cdot \exp\left(\frac{5\epsilon\tau B}{12}\right) \cdot \exp\left(\frac{\tau A}{\sqrt{5}}\right) \cdot \exp\left(\frac{5\epsilon\tau B}{12}\right) \\ &\quad \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right] \cdot \exp\left(\frac{\epsilon\tau B}{12}\right) \\ &= \exp\left[\tau F + \epsilon^2\tau^3\left(\frac{13-5\sqrt{5}}{288}\right)[B, B, A] + O(\epsilon\tau^7)\right] \end{aligned}$$

McLachlan (1995) has independently derived similar solutions.

Unlike the 4th-order algorithm of Forest and Ruth (1990) and the 6th-order integrators of Yoshida (1990), the algorithms above contain no substeps that move in the opposite direction to the main integration. An additional solution exists for each of equations (9), (11) and (12), however these have error terms with larger numerical coefficients than the integrators we show here.

The same method can be used to generate a pseudo-8th-order integrator and so on. Each higher order will require only 2 more substeps than the previous one, since only one more commutator needs to be eliminated in each case. For example, to create a pseudo-8th-order integrator requires the elimination of the $[A, A, A, A, A, A, B]$ term in addition to those that are absent from the pseudo-6th-order case. However, depending on the system to be integrated, there will come a point at which the $\epsilon^2\tau^3$ error term becomes the most important. In principle, one could devise another set of integrators that eliminates terms in $\epsilon^2\tau^m$ for small m , in addition to terms in $\epsilon\tau^m$. However, achieving each new order will generally require the elimination of more than one commutator term, so that these integrators increase in complexity much more rapidly than those described here.

Murison and Chambers (1999) have independently derived the two 4th-order integrators above, among others, using a symbolic algebra package. Further results from that approach will follow in another paper. We note that the pseudo-order algorithms can be adapted to use independent timesteps for each planet (*c.f.* (Saha and Tremaine 1994)), or to include close encounters ((Duncan *et al.* 1998; Chambers 1999)).

4. Numerical Comparisons

In this section, we test the pseudo-4th and 6th order integrators derived in Section 3 against the well-known 2nd and 4th-order symplectic algorithms. We use the “mixed-variable” method of Wisdom and Holman (1991), in which the Hamiltonian is divided into a Keplerian part, H_K , and an interaction part, H_I . Under H_K , each object moves on an unperturbed Keplerian orbit about the central body. Under H_I , each object remains fixed while receiving an impulse due to the gravitational perturbations of all the other objects except the central body. As suggested by Wisdom and Holman, we use Jacobi coordinates rather than barycentric coordinates. The integrations themselves were carried out using a modified version of the *Mercury* integrator package ((Chambers and Migliorini 1997)).

The pseudo-order integrators require that the ratio $\epsilon = H_I/H_K \ll 1$. In our first test, we integrate the orbits of the 4 inner planets of the solar system in the absence of the outer planets. In this case $\epsilon \sim 10^{-5}$. Figure 1 shows the results of a 10000-year integration using the conventional 2nd and 4th-order symplectic integrators, S2B and S4B, and the pseudo-order integrators S4B* and S6B*. For each integration, the maximum relative energy error is shown as a function of the step size.

For the 2nd and 4th-order integrators, the maximum energy error is roughly proportional to τ^2 and τ^4 respectively, where τ is the timestep. This is what we would expect to find. For the

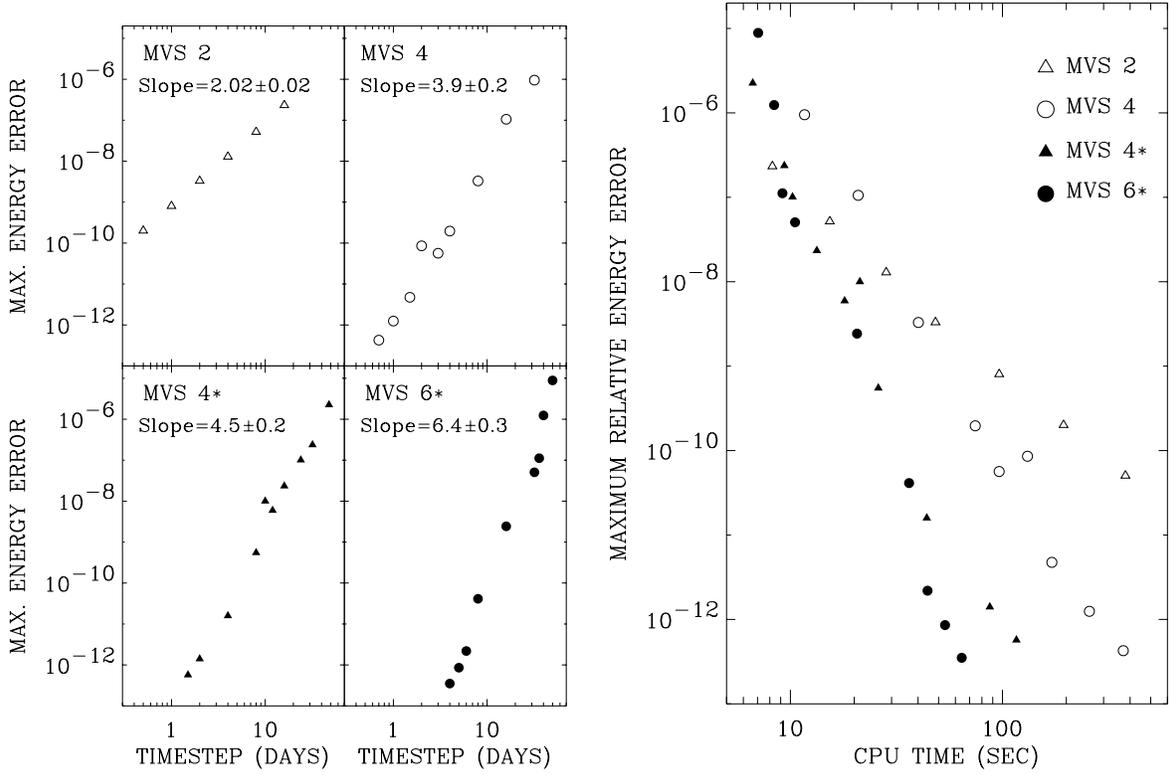


Fig. 1.— Maximum relative energy error versus step size for a 10000-year integration of the 4 terrestrial planets using various symplectic integrators.

Fig. 2.— Maximum relative energy error versus CPU time for a 10000-year integration of the 4 terrestrial planets using various symplectic integrators.

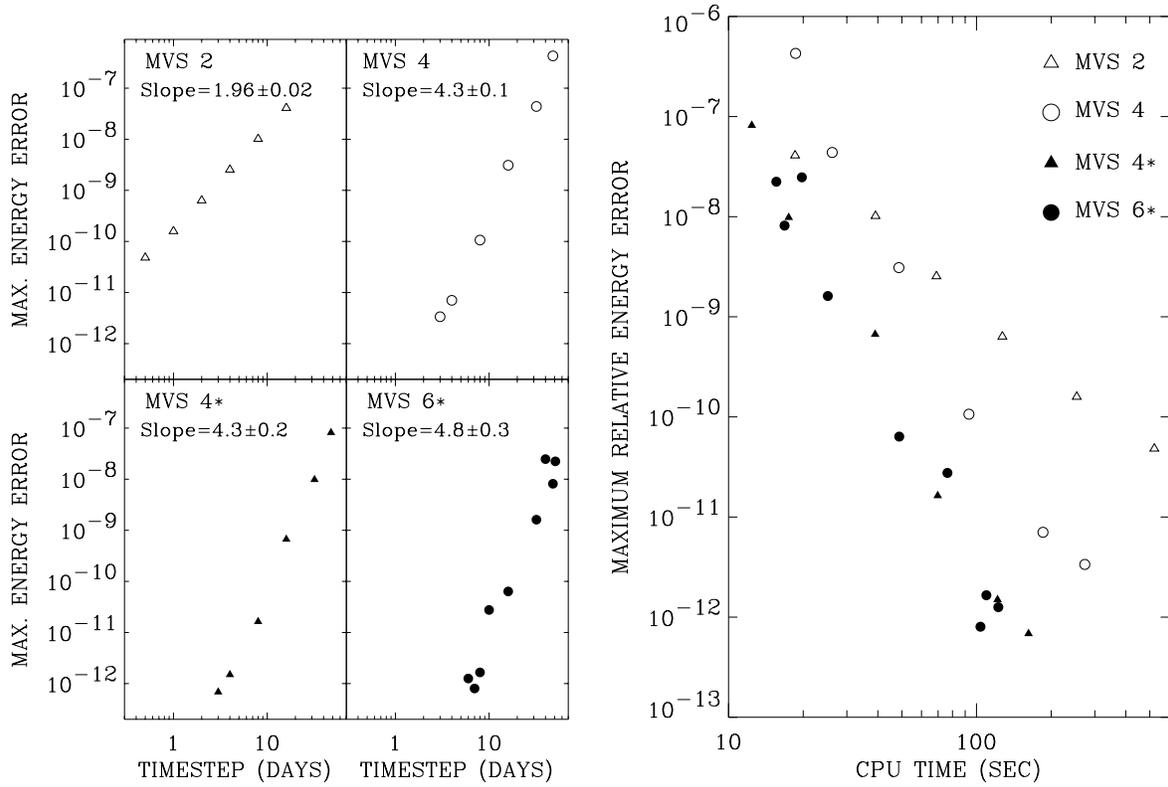


Fig. 3.— Maximum relative energy error versus step size for a 10000-year integration of the 9 planets using various symplectic integrators.

Fig. 4.— Maximum relative energy error versus CPU time for a 10000-year integration of the 9 planets using various symplectic integrators.

pseudo-4th and 6th-order integrators, the maximum energy error varies as τ^4 and τ^6 . That is, they behave as 4th and 6th-order integrators, as we anticipated, despite the fact that they contain error terms of lower order in the timestep.

Using the mean relative energy error per integration instead of the maximum error gives results similar to Figure 1. The corresponding slopes are 2.10 ± 0.05 for S2B, 3.9 ± 0.3 for S4B, 4.6 ± 0.3 for S4B* and 6.4 ± 0.4 for S6B*.

Figure 2 shows the amount of CPU time required for the integrations shown in Figure 1. For energy errors of 1 part in 10^6 or 10^8 there is not much to choose between the four algorithms. For higher levels of accuracy, S4B outperforms S2B. However, the pseudo integrators S4B* and S6B* do even better. At an accuracy of 1 part in 10^{10} , they are roughly an order of magnitude faster than the conventional second-order integrator, and 3 times faster than the 4th-order integrator. For accuracies of better than 10^{-11} , S6B* shows greater performance than S4B*.

The pseudo-4th order integrator is more efficient than the real 4th-order integrator for two reasons. It requires fewer substeps per time step, and it has a slightly smaller leading error term.

As a more interesting test, we integrated the whole planetary system (Mercury to Pluto) for 10000 years. Figure 3 shows the energy-error results of these integrations. The behaviour of S2B, S4B and S4B* is similar to that for the integrations of the terrestrial planets. However, the energy error for S6B* varies roughly as τ^5 rather than τ^6 . It is not obvious why this should be, although the difference from the terrestrial-planet integration (Figure 1) is presumably due to the fact that ϵ is two orders of magnitude larger in this case.

Figure 4 shows the CPU time required for the integrations of the 9 planets. The results are similar to the integration of the inner planets, except that S6B* has only a marginal advantage over S4B* at the highest levels of accuracy.

Since writing the original draft of this manuscript, we have become aware of the symplectic corrector method of Wisdom *et al.* (1996), which substantially improves the efficiency of the second-order symplectic integrator. We present the pseudo-order integrators as an alternative strategy for designing accurate algorithms. It is possible to devise other symplectic correctors using the same approach we use in Section 3 to design the integrator kernel: that is, by considering the dependence of the resulting error terms on ϵ as well as τ ((Mikkola 1997; Rauch and Holman 1999)). Finally we suggest that it may be possible to design symplectic correctors to improve the performance of pseudo-order algorithms, since the pseudo-order methods exhibit similar high-frequency oscillations in energy error to the second and 4th-order symplectic integrators (see Figure 5).

In summary, we conclude that the new pseudo-order integrators outperform the widely-used 2nd and 4th-order algorithms at all reasonable values of the energy error, for problems involving a dominant central mass.

Research at Armagh Observatory is grant-aided by the Dept. of Education, Northern Ireland.

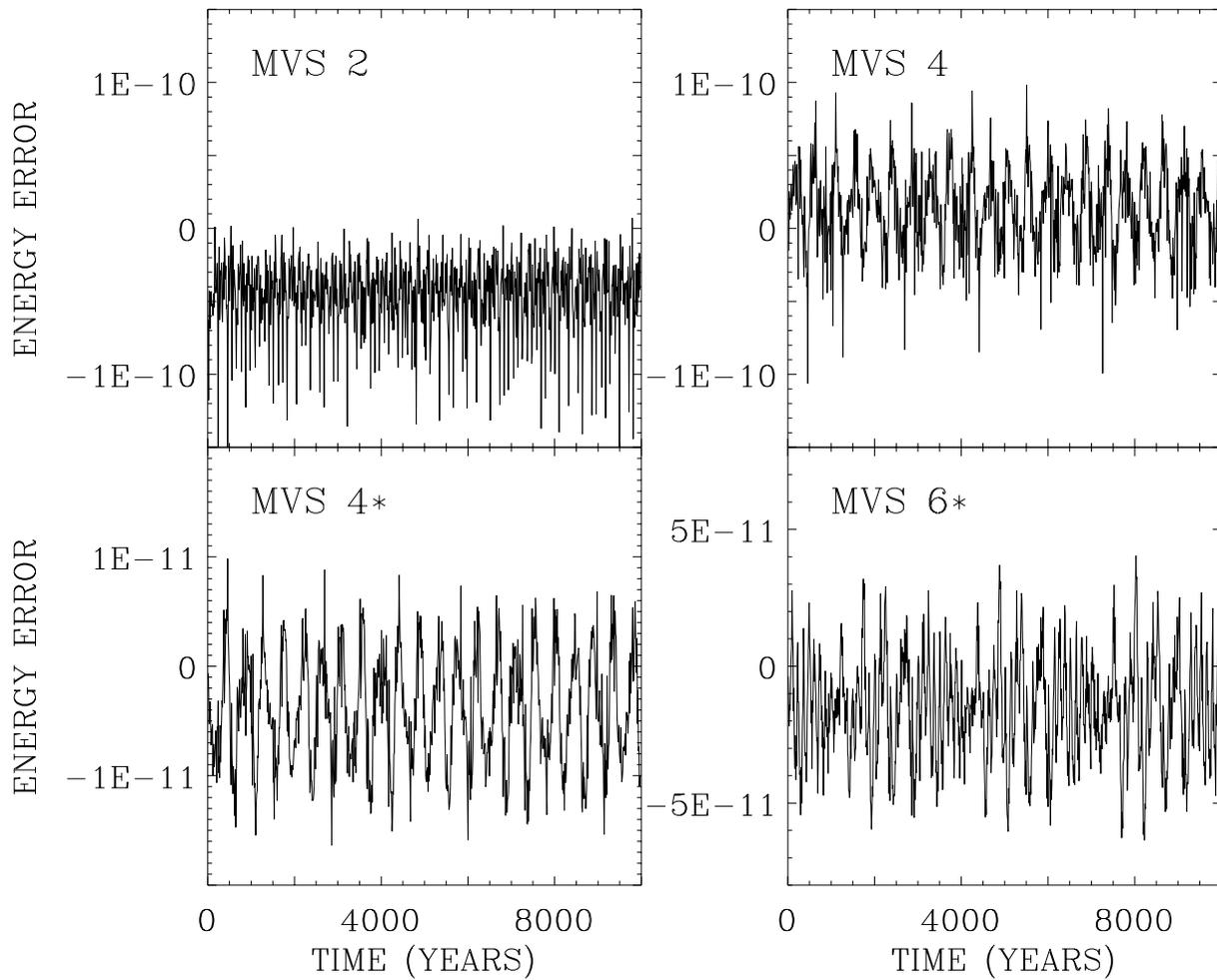


Fig. 5.— Relative energy error versus time for 10000-year integrations of the 9 planets using various symplectic integrators.

The test integrations described in this paper were carried out using computers purchased on a PPARC research grant.

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Figure Captions

Figure 1: Maximum relative energy error versus step size for a 10000-year integration of the 4 terrestrial planets using various symplectic integrators.

Figure 2: Maximum relative energy error versus CPU time for a 10000-year integration of the 4 terrestrial planets using various symplectic integrators.

Figure 3: Maximum relative energy error versus step size for a 10000-year integration of the 9 planets using various symplectic integrators.

Figure 4: Maximum relative energy error versus CPU time for a 10000-year integration of the 9 planets using various symplectic integrators.

Figure 5: Relative energy error versus time for 10000-year integrations of the 9 planets using various symplectic integrators.