Resonance Overlap and the Applicability of the Lyapunov Exponent Relation

Dynamical Stability in the Outer Asteroid Belt

Marc A. Murison

U.S. Naval Observatory

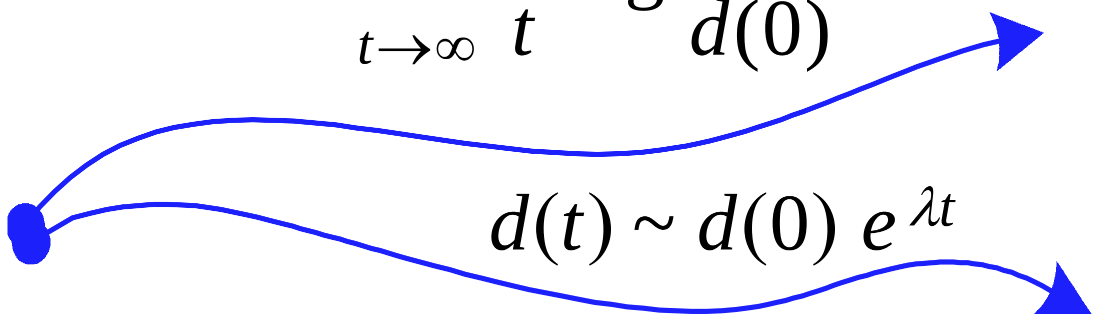
Washington, DC

murison@riemann.usno.navy.mil

Introduction

- Lyapunov exponent λ is a measure of orbit separation:

$$\lambda \equiv \lim_{t \to \infty} \frac{1}{t} \log \frac{d(t)}{d(0)}$$



$$d(t) \sim d(0)\, e^{\lambda t}$$

  where d(t) is distance in phase space.

- $\lambda > 0$ indicates chaos

- Magnitude of λ is a measure of severity of chaotic motion.

- **Question**: how can we relate a "Lyapunov timescale"

$$T_L \equiv \frac{1}{\lambda}$$

  to a dynamical stability timescale?

Introduction

- Soper et al. (1990) discovered empirically a *Lyapunov exponent relation*:

$$\log \frac{T_e}{T_0} = a + b \log \frac{T_L}{T_0}$$

where T_e is the "event" timescale (time to orbit crossing, collision, or ejection) and a, b, and T_0 are constants.

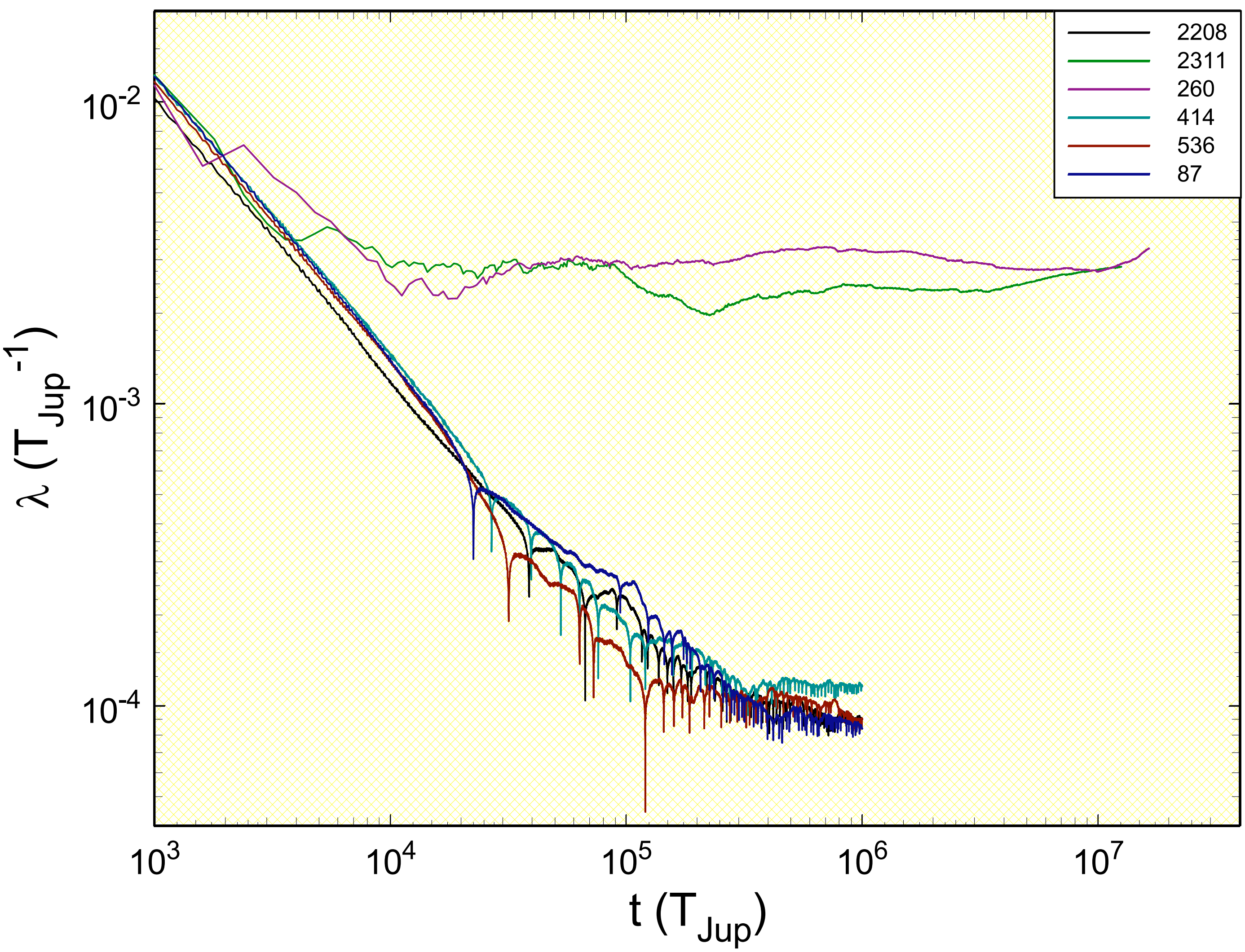
- For a large variety of dynamical systems,

$$0.4 \le a \le 2 \quad \text{and} \quad 1.4 \le b \le 1.9$$

- Integrating a dynamical system (e.g. the solar system) to T_e is normally very expensive, while integrating just to determine T_L is not (by ~3-5 orders of magnitude). Hence the relation is potentially a powerful prediction mechanism.

Introduction

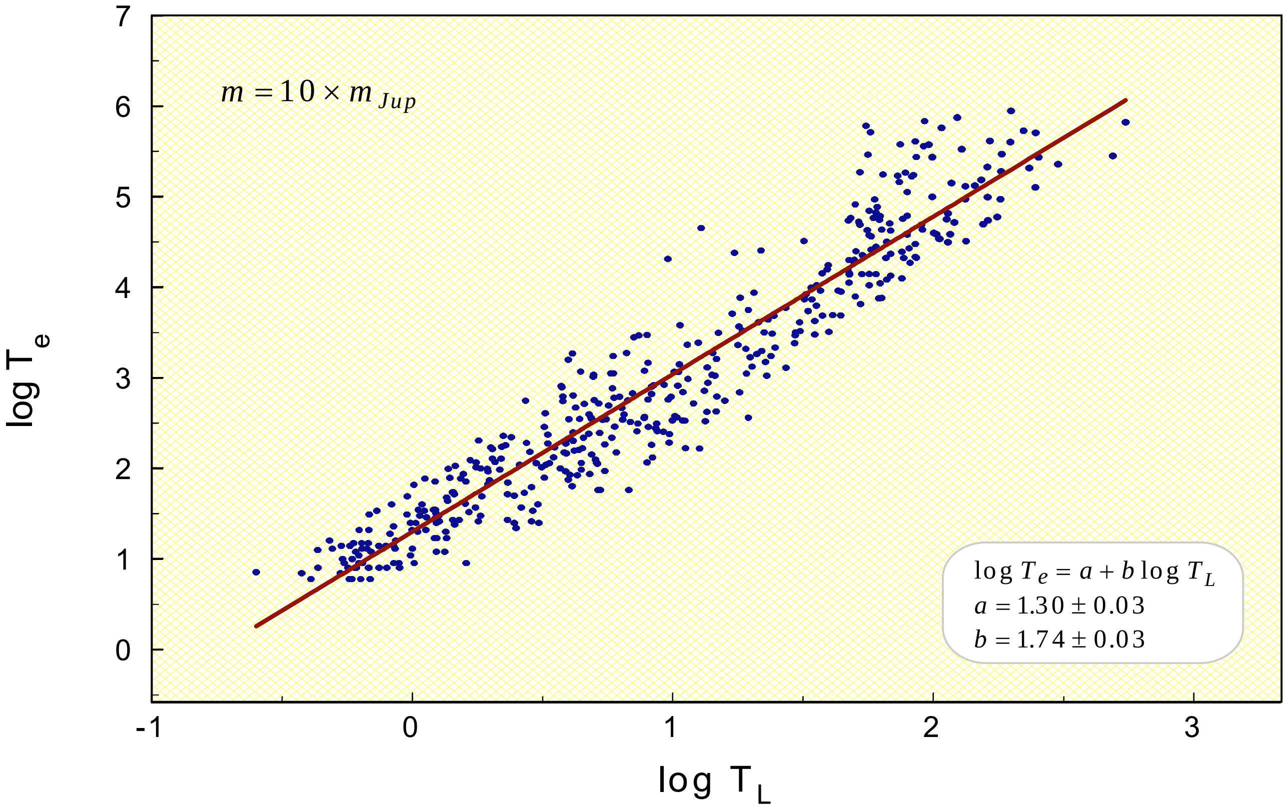
- Typical behavior of the Lyapunov exponent over time for chaotic orbits:



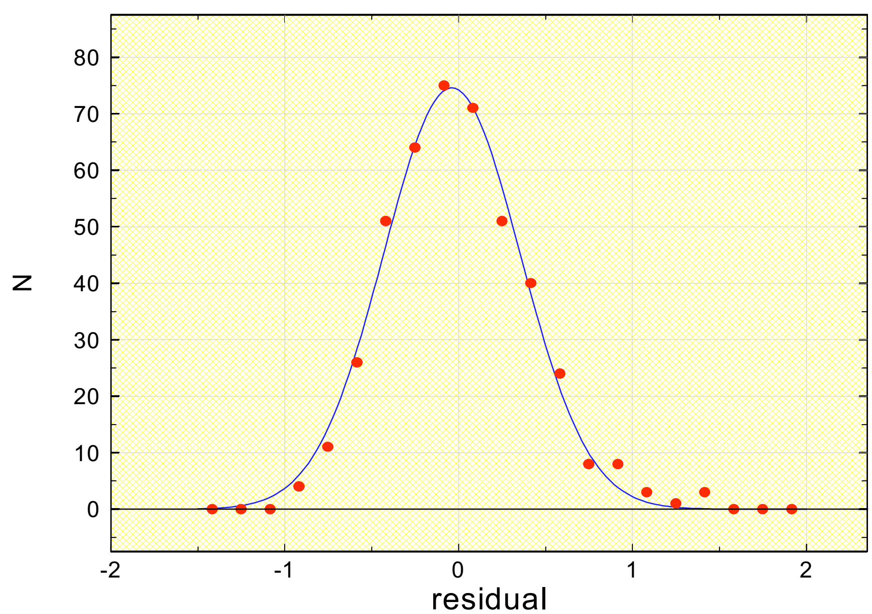
Here we have λ(t) for 6 outer belt asteroids. All exhibit strong chaotic motion, while two are extremely chaotic. Time is measured in Jovian years (11.86 yr).

Introduction

- Lyapunov exponent relation is robust:



- Residuals in $\log(T_e)$ are well-fit by a Gaussian.



Applications

To date, the relation has been applied or verified in the following dynamical settings:

- Asteroids interior to Jupiter (Soper et al. 1990, Lecar et al. 1992)
- Escape from the smaller mass of a binary system (Lecar et al. 1992)
- Objects between Jupiter and Saturn (Lecar et al. 1992)
- Kuiper belt objects (Levison and Duncan 1993)
- Outer solar system and short-period comet reservoir (Holman and Wisdom 1993)
- Hilda group asteroids (3:2 resonance) (Franklin et al. 1993)
- 2:1 Kirkwood gap (Franklin 1994)
- Outer asteroid belt (Murison et al. 1994)

Problems with the Relation

- Normalizing timescale T_0
  - What to use for T_0 when more than one perturber?
  - rescales a
- Still can need *long* integrations: 3σ in a power law exponent covers a lot of territory:

$$\frac{T_e}{T_0} = A \cdot \left(\frac{T_L}{T_0} \right)^b \quad \text{where} \quad \log A \equiv a$$

- Calibration: since we don't understand the theoretical underpinnings, we must calibrate the relation for each new dynamical situation to make sure we're on safe ground. This can be very expensive.
- Evidence of "anomalously" long-lasting orbits with large Lyapunov exponents (e.g. Murison et al. 1994, Whipple 1995, and this paper).
- **When is the relation applicable?**

Digression: Resonance Overlap Regime

- The "resonance overlap" regime (i.e., strong perturbations) is where we find global chaos.
- Dynamics governed by a FP diffusion equation for the distribution function in action space:

$$\frac{\partial N(I,t)}{\partial t} = \frac{\partial}{\partial I}\left[D \frac{\partial N(I,t)}{\partial I} \right] - \frac{N(I,t)}{Te}$$

- Numerical experiments (Konishi 1989) relate the diffusion coefficient D to λ:

$$\log D \approx \tilde{a} + \tilde{b} \log \lambda$$

- Varvoglis & Anastasiadis (1996) combine these to retrieve the Lyapunov exponent relation:

$$\log T_e \approx a + b \log T_L$$

  - Problem: the power law behavior is built into the diffusion equation from the outset (2nd term on rhs), so this argument is suspect.
  - However, this supports the numerical evidence that the relation is valid in the overlap regime for a large variety of dynamical systems.

Digression: Isolated Resonance Regime

- In the isolated resonance regime (i.e., nonoverlapping resonances or "small" perturbations), we must use Lévy statistics to describe random (i.e., diffusive) motion, rather than the usual Brownian motion that most of us are used to.
- Lévy statistics is used to describe fractal (scale invariant) random processes.
- Mean-squared particle displacement is

$$\left\langle r^2(t) \right\rangle \sim t^\gamma$$

γ = 1 Brownian, γ = 2 ballistic, γ = 3 turbulent diffusion

- Hamiltonian systems: $1 < \gamma < 2$. (This neatly explains Hamiltonian intermittency.)
- Motion is governed by a Fokker-Planck-Kolmogorov diffusion equation.
- See Shlesinger et al. (1993), Klafter et al. (1996).
- It is NOT clear if the Lyapunov exponent relation holds in this regime. Solving the FPK diffusion equation is very difficult in this case.

New Results – Outline

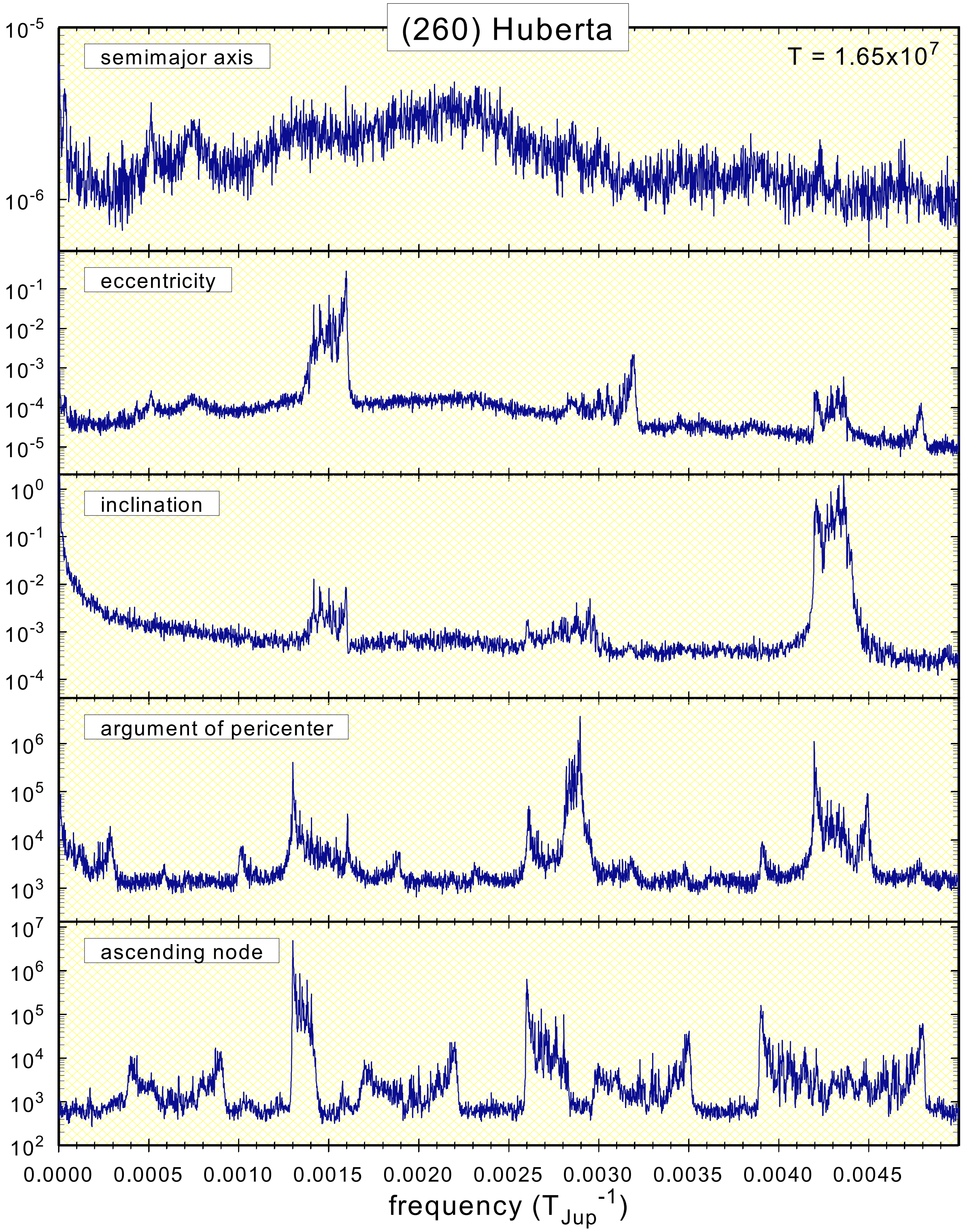
- 3D elliptic restricted three-body integrations
- (1) Severe chaos ("anomalously large" to some)
  - Examples are easy to find in the outer asteroid belt.
- (2) Resonance overlap threshold
  - Transition is fairly sharp.
- (3) Stable and unstable regions
  - Imaging λ as a function of (a_0,e_0) reveals complex structures.
- (4) Correlation between λ and mean motion resonances
  - The Lyapunov exponent increases sharply in magnitude across high-order mean motion resonances.
  - First seen by Murison et al. (1994).

New Results – (1) Severe Chaos

- Orbits are stable over very long timescales, despite very large λ.
  - (260) Huberta for >500 Myr
  - (2311) El Leoncito for >210 Myr
  - "3σ" outer belt asteroids from Murison et al. 1994?
- Real-life examples are easy to find in the outer asteroid belt.
  - 21 of 25 with T_L < 60,000 yr
- Characteristics:
  - Very large λ
  - Very broad power spectral features
  - Confinement near or between porous hypersurfaces

New Results – (1) Severe Chaos

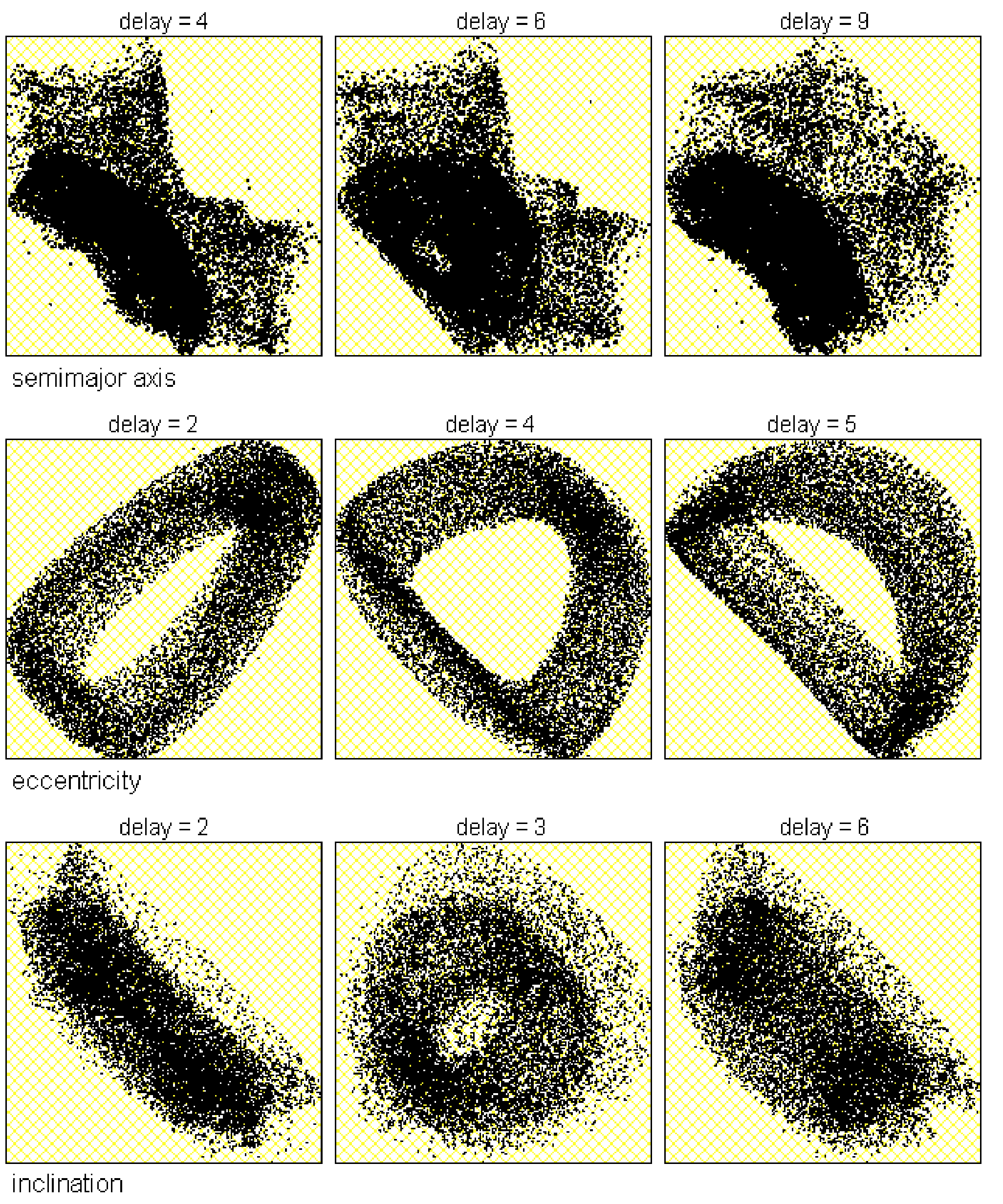
- *Extremely* broad power spectral features



New Results – (1) Severe Chaos

● Confinement near hypersurfaces

Each panel below is a time-delay phase space reconstruction from semimajor axis, eccentricity, and inclination time series of a single integration. The structures are topologically equivalent to the actual phase space structures. Each plot contains 20,000 data points.



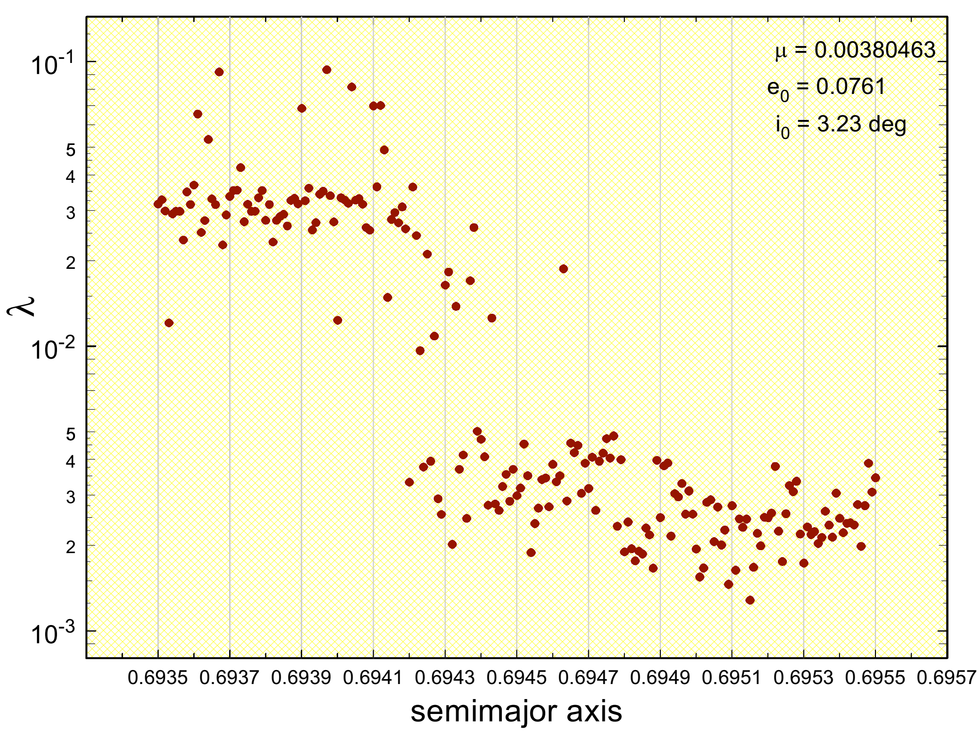
(414) Liriope

New Results – (2) Overlap Threshold

- Previous results (e.g. Murison et al. 1994) were in the overlap regime!
- Transition between overlap/nonoverlap is abrupt.
- Location of transition point agrees with Tremaine's approximation

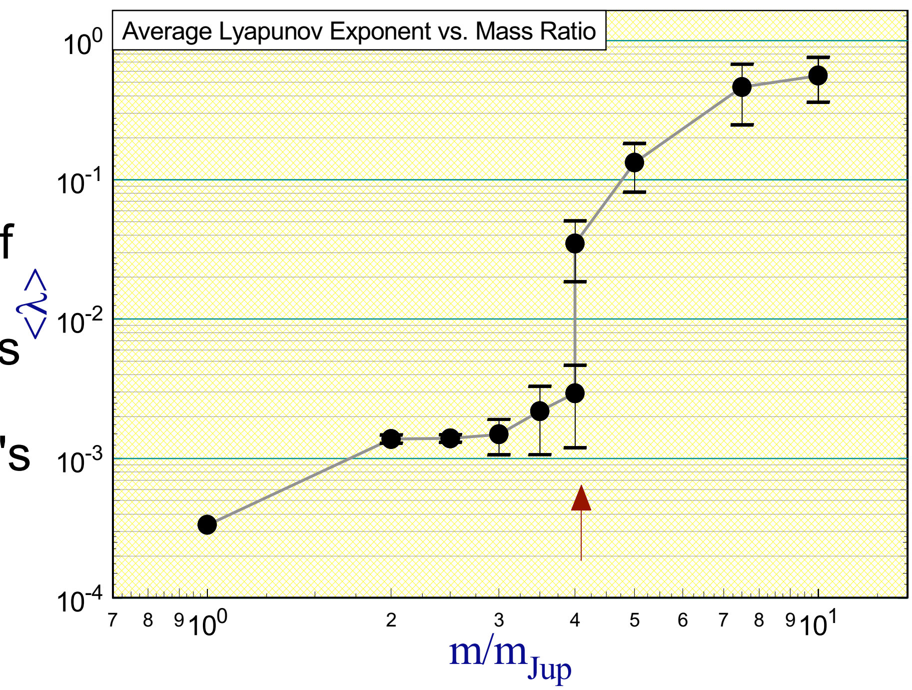
$$\Delta a \approx 1.49\,\mu^{2/7}$$

Here is a plot of λ vs. initial semi-major axis for a set of fictitious asteroids, illustrating the sharpness of the transition between the two regimes.



New Results – (2) Overlap Threshold

This is a plot showing the behavior of λ as a function of Jupiter/Sun mass ratio. Each data point represents an average of hundreds of separate orbit integrations. The arrow marks the transition point predicted by Tremaine's approx- imation. The data does not agree with the earlier, similar result due to Wisdom.



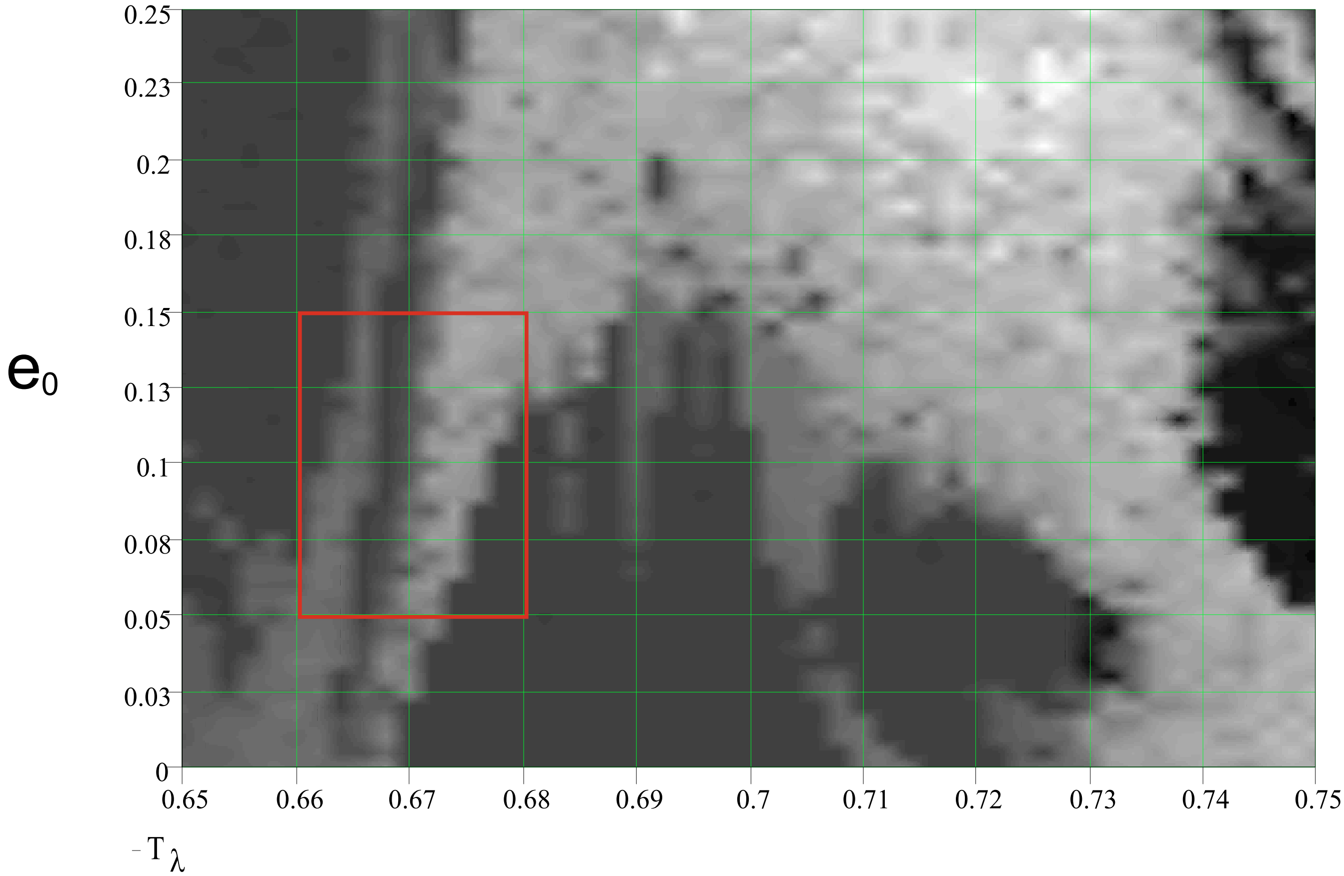
- Evidence that relation is valid in overlap regime but more complicated in isolated resonance regime:
  - abrupt change in λ across threshold
  - bounded chaos examples (cf. confinement to hypersurfaces)
  - Varvoglis diffusion argument

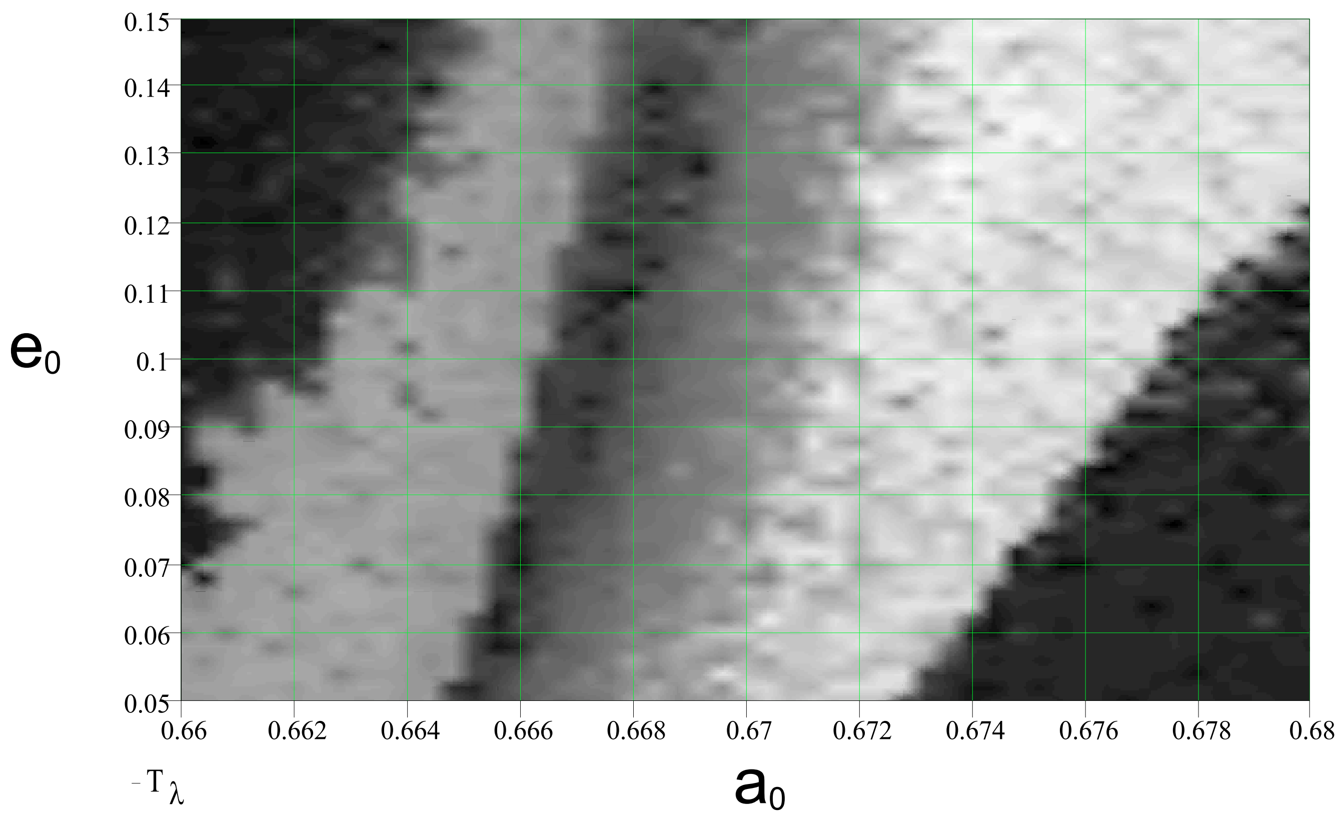
New Results – (3) Stable Regions

- Structures are smooth functions of perturbing strength (mass ratio).
- Structures formed by families of periodic orbits (characteristic curves).
- Structures are fractal or fractal-like (Murison 1988, 1989).
- <a> and <b> from Lyapunov relation are functions of
  - e_p
  - mass ratio μ (i.e., perturbing strength)
  - (a_0, e_0)

The following two plots are images of λ as a function of the initial semimajor axis and the initial orbital eccentricity of 51x51 orbits. In the upper image, orbits were integrated to 10^4 Jovian years. The lower image is a magnification of a portion of the upper image. In the lower image, orbits were integrated to 10^5 Jovian years.

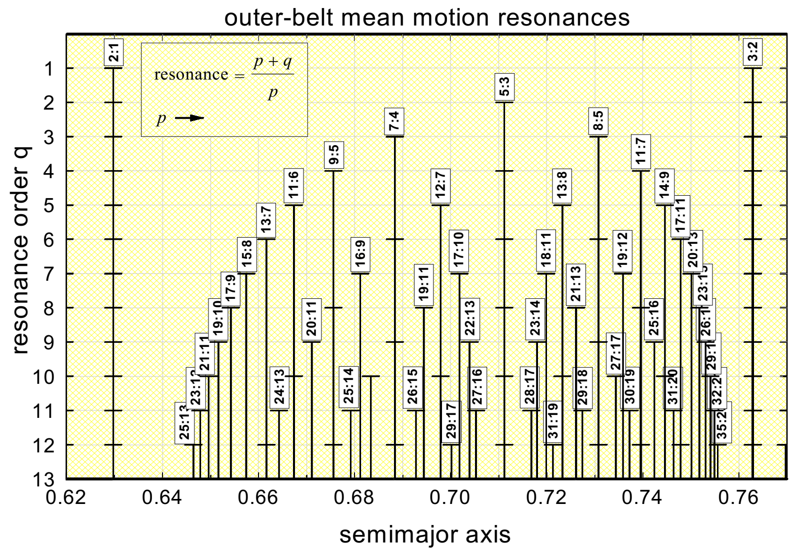
New Results – (3) Stable Regions



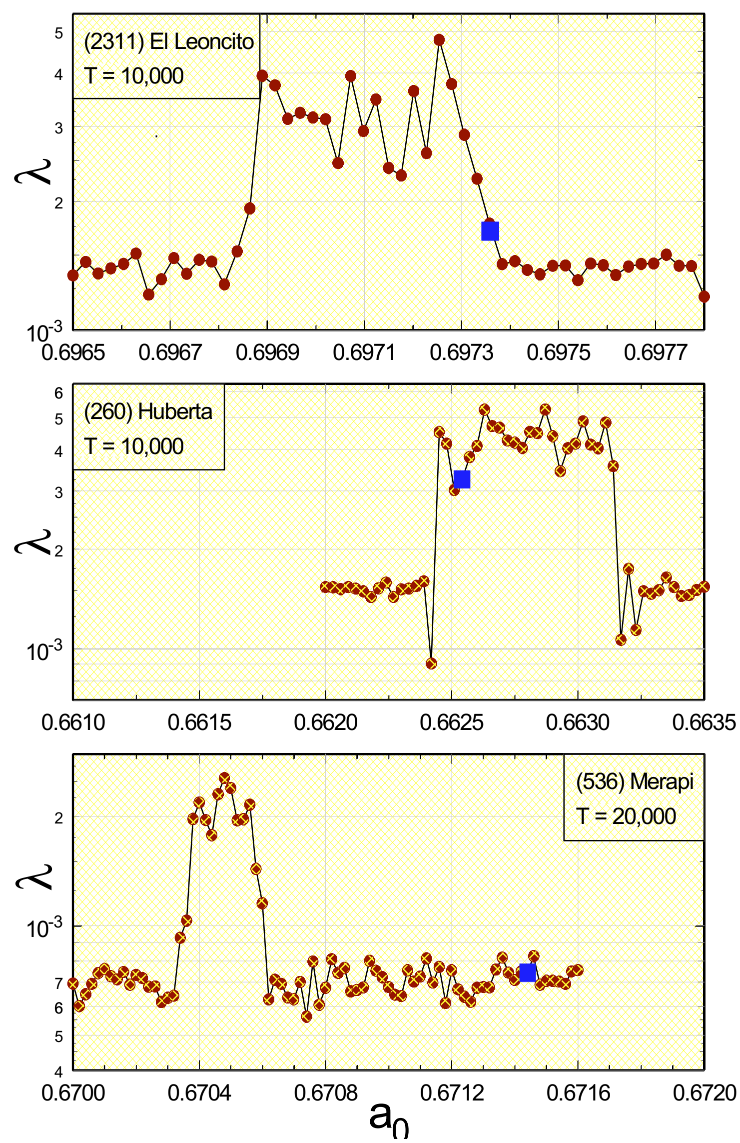


New Results – (4) Resonances

- Large values of λ associated with high order mean motion resonances



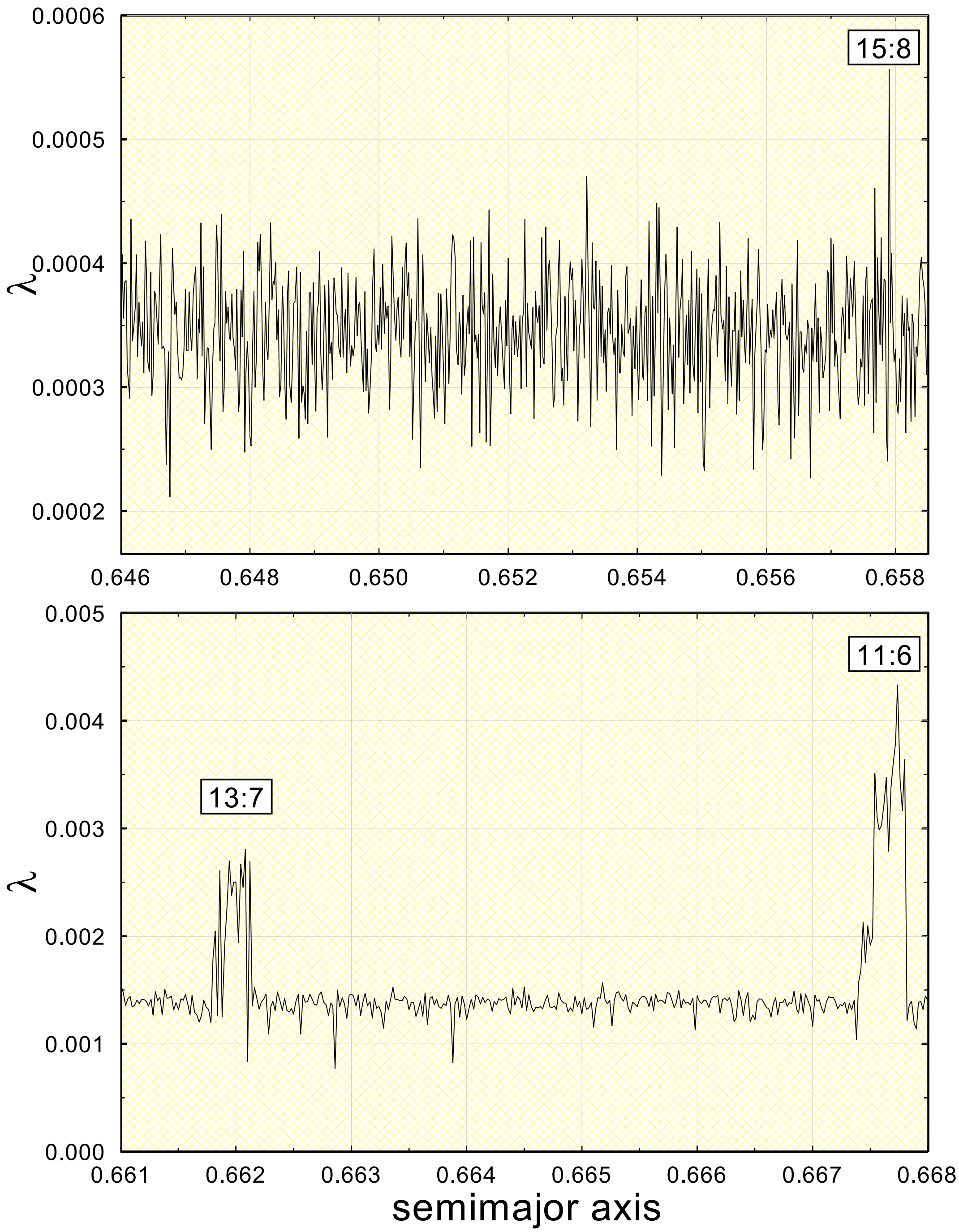
The top plot shows the mean motion resonance structure of the outer belt. It is complete up to 12th order resonances.



The next three plots show λ as a function of initial semimajor axis in the vicinities of three outer belt asteroids. The blue squares mark the initial semimajor axes of these objects. Each region of increased λ corresponds to a high-order mean motion resonance.

New Results – (4) Resonances

These 2 panels are a scan of λ across initial semimajor axis in a small section of the outer asteroid belt (but larger than the snippets in the previous panels). The detected resonances are marked. Note the change of ordinate scale in the upper panel.



Conclusions

- The Lyapunov exponent relation is applicable in regions where phase space barriers have been destroyed by strong perturbations.
  - Resonance overlap regime
  - "Global" sea of chaos
- It is still unclear if or how the relation applies in the isolated resonance (weak perturbation) regime.
- Overlapping resonance transition is abrupt.
- Orbits with very large λ can be common in the isolated resonance regime. This is certainly the case in the outer asteroid belt.
- In the isolated resonance regime, chaotic motion is confined between porous, nested hypersurfaces.
- Objects in high-order mean-motion resonances exhibit significantly stronger chaos.

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