Harnessing Radiation Torques to Drive the Precession of a Spin-Stabilized Spacecraft

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Introduction

- U.S. Naval Observatory wants to launch FAME
  - Fast Astrometric Mapping Explorer
  - Hipparcos-style survey instrument
  - ~100,000 km orbit
  - < 50 μas at 9th magnitude
- Scanning spacecraft
  - Spin-axis stabilized
  - Use precession to cover sky
  - Need thruster firings to drive precession
  - This can play havoc with data reduction
    - Need to "tie" together data
      - across thruster discontinuities
      - from one spin period to the next

Scan Pattern on Sky

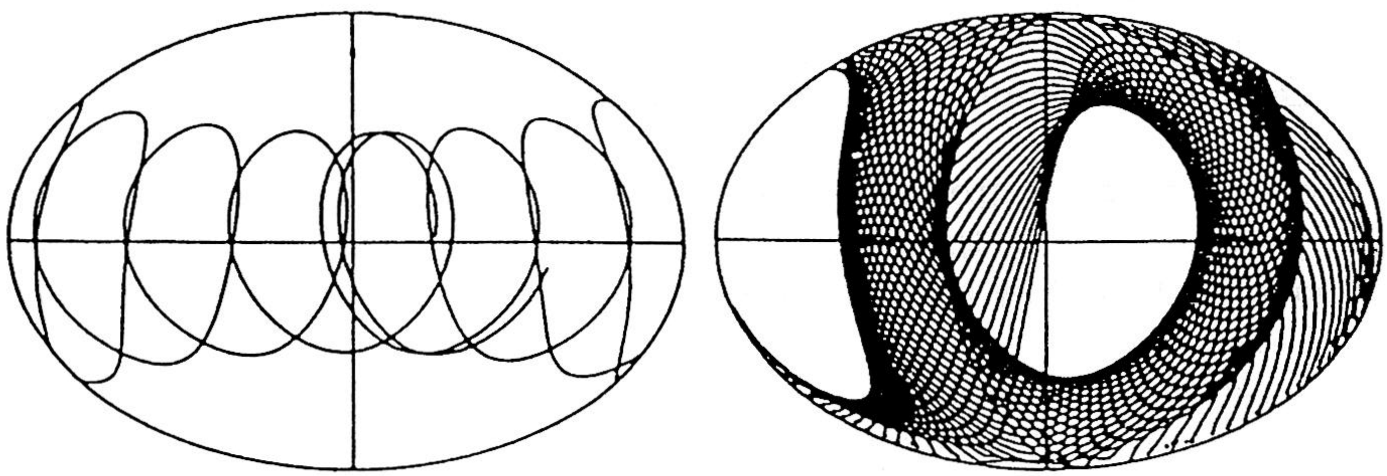
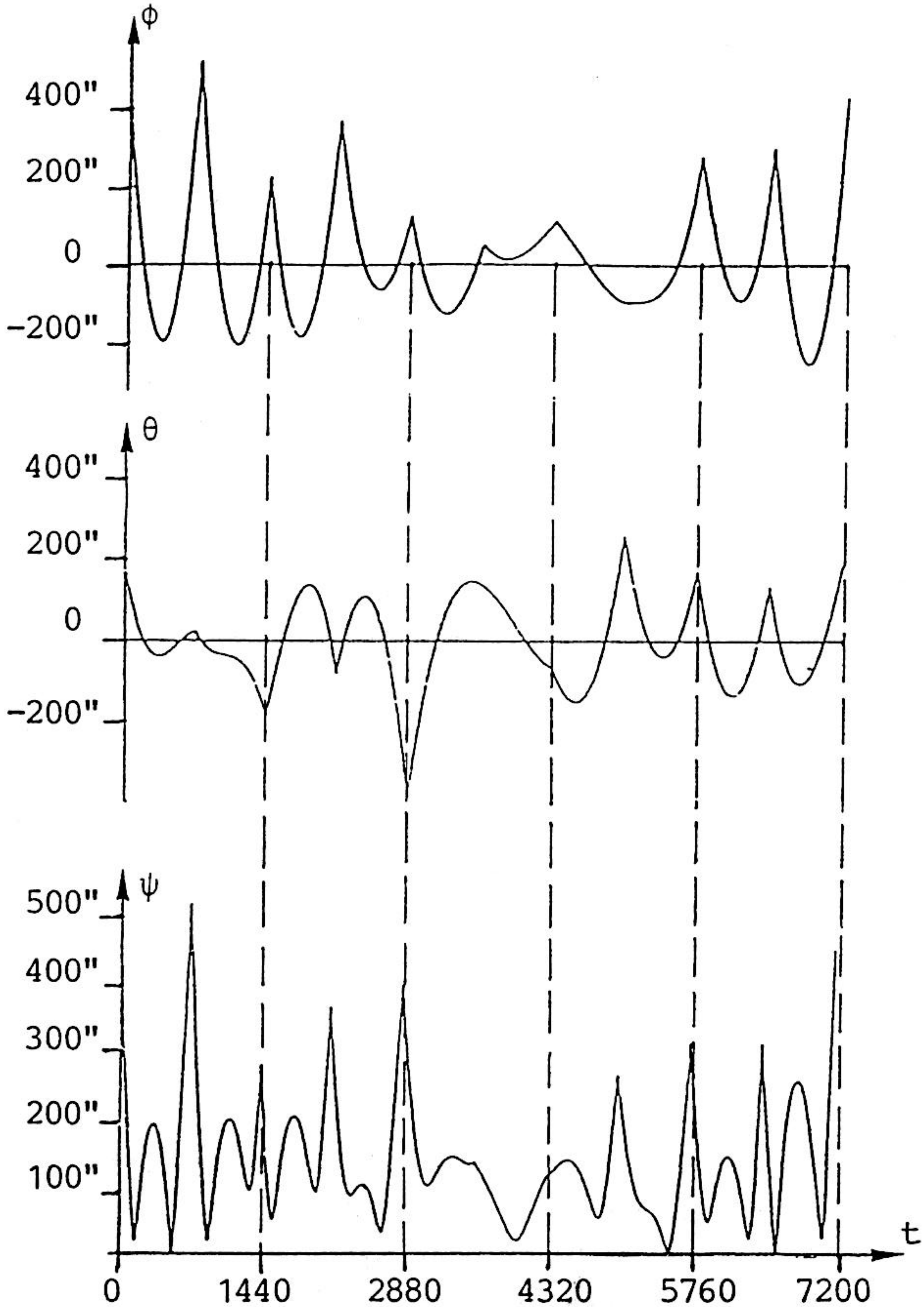


Fig. 8.5. Hipparcos nominal scanning law in ecliptic coordinates - *Left*: motion of the satellite axis in one year - *Right*: part of the sky scanned in 70 consecutive days

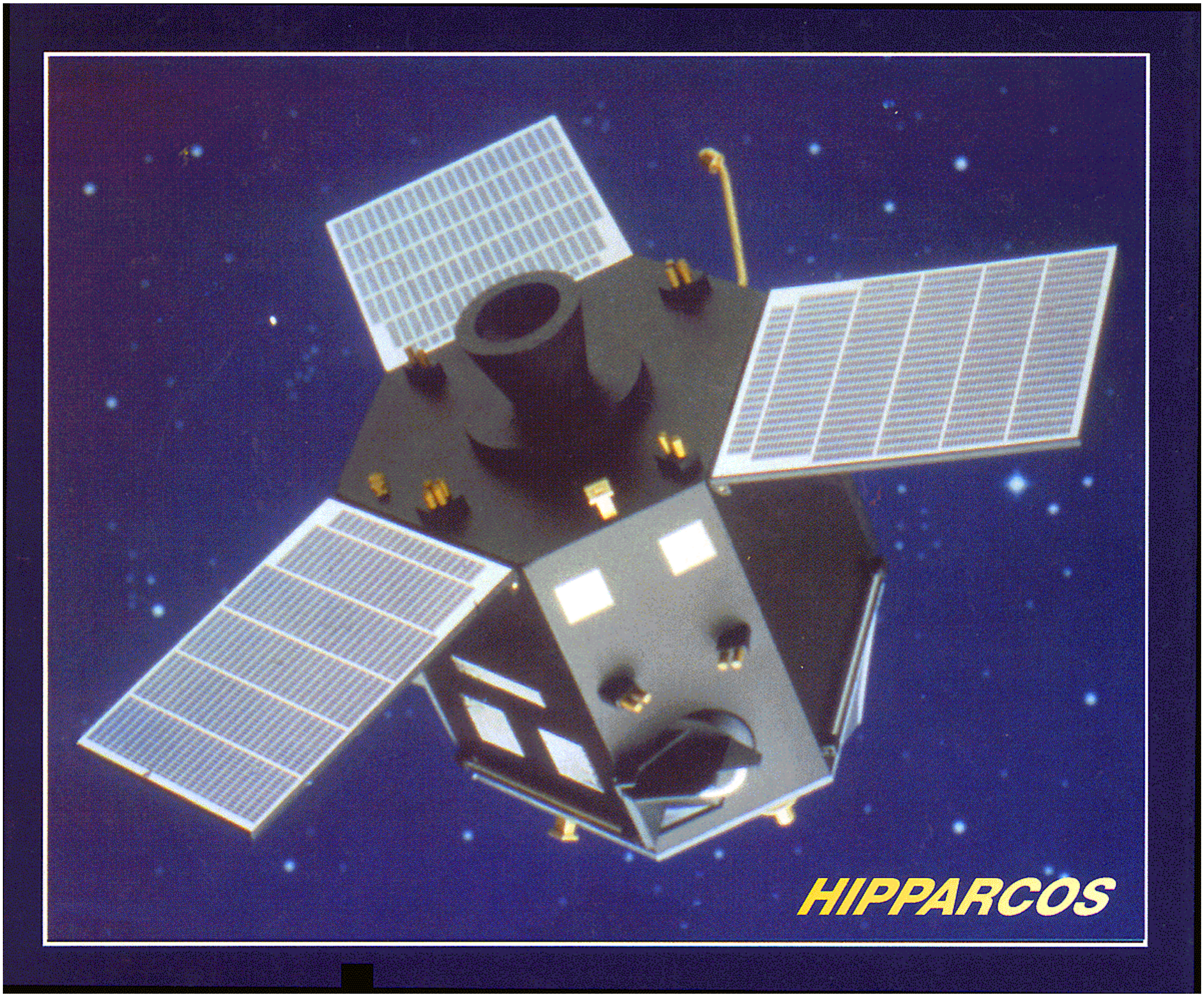
Hipparcos Attitude Corrections



Introduction (continued)

- CfA group: covariance studies
  - gain a factor of ~5 in astrometric accuracy in going from 6 per rotation to once per rotation
  - gain an order of magnitude in extreme case of no thruster events
- **Can we do away with (most) thruster firings?**
- Maybe!
  - high orbit (~100,000 km)
  - solar shield (needed for thermal control)
  - use radiation pressure to drive precession and control precession rate

Hipparcos



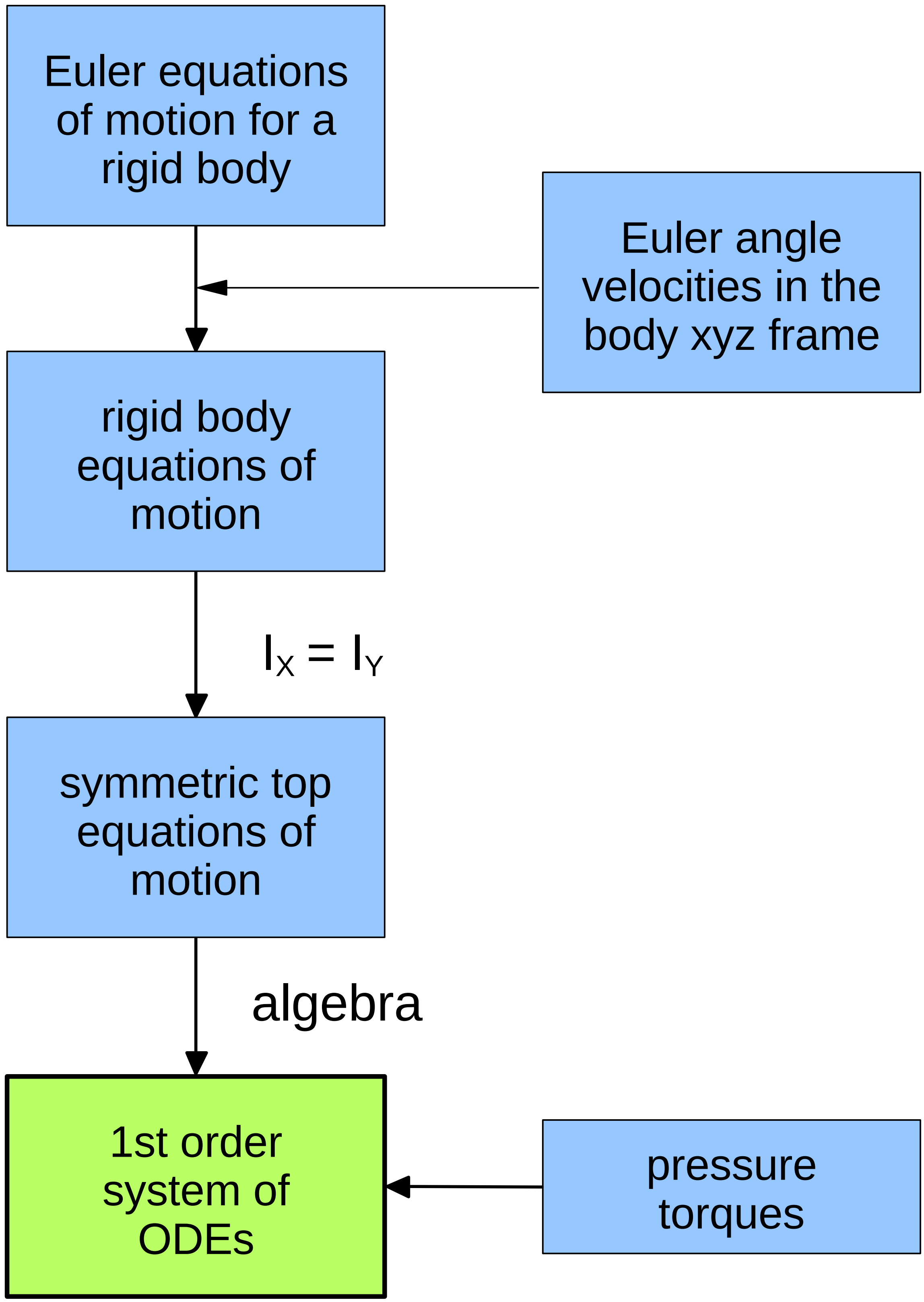
This Study

- Task: characterize precession of spin axis as driven by radiation pressure.
  - cylindrical spacecraft
  - shield: frustum of a cone, albedo A_C
  - "flat top", albedo A_T
- Curiosity: interesting dynamics?

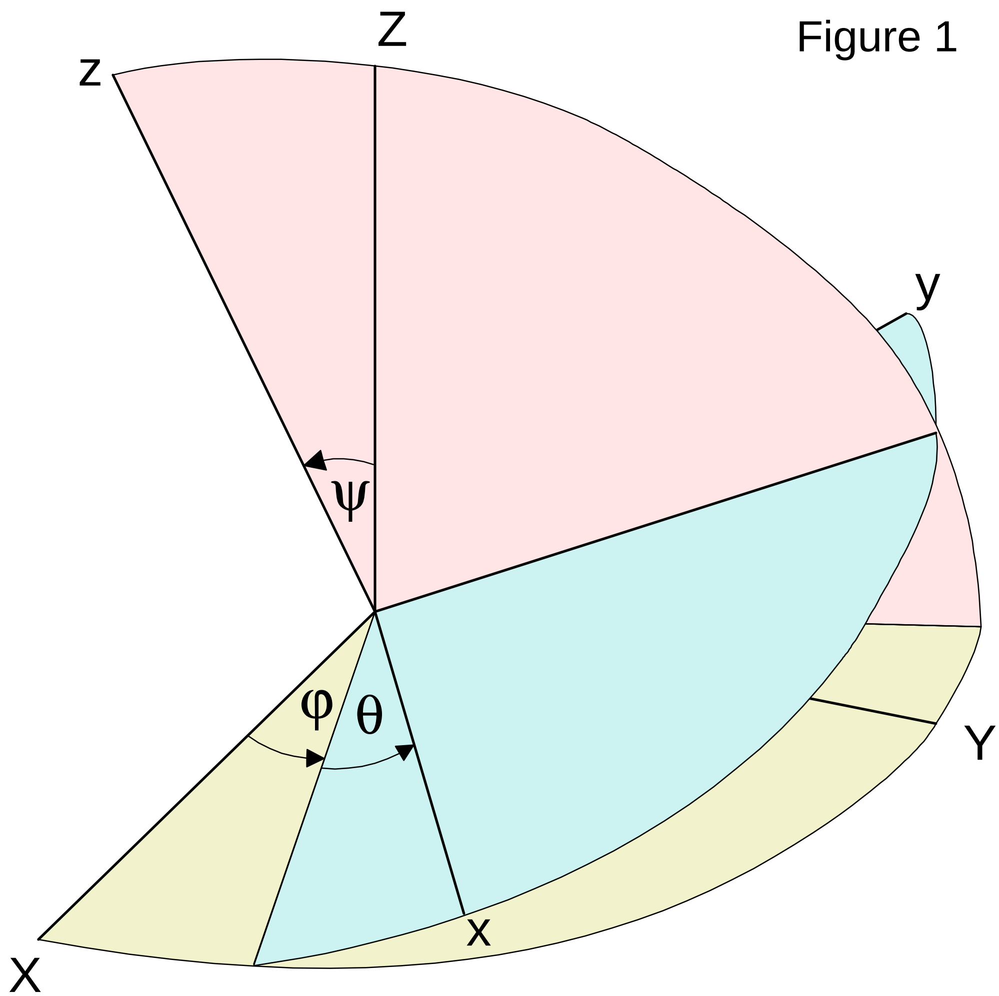
  $\Rightarrow$ derive full equations of motion

  $\Rightarrow$ numerical program

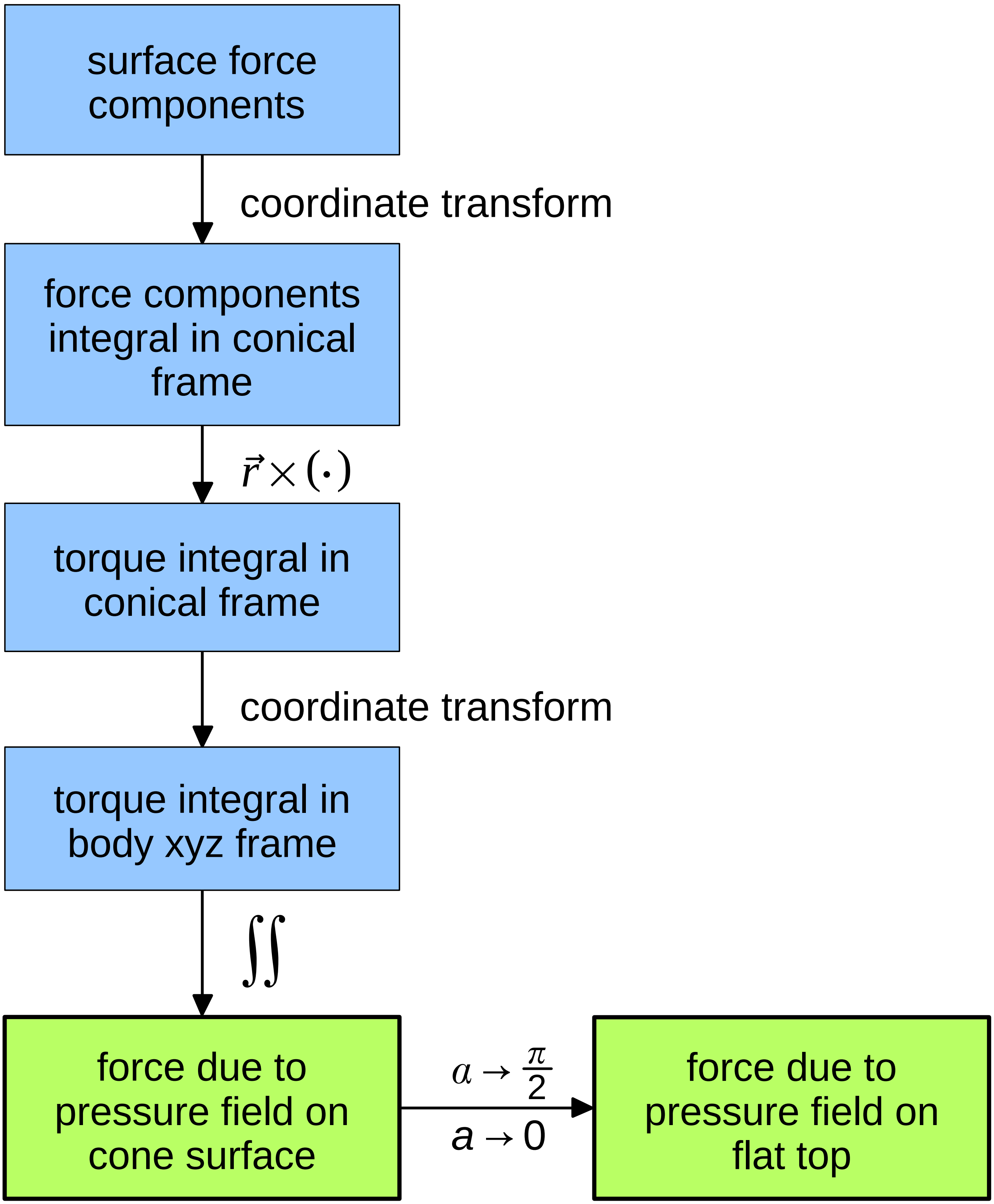
1. Equations of Motion



Euler Angles

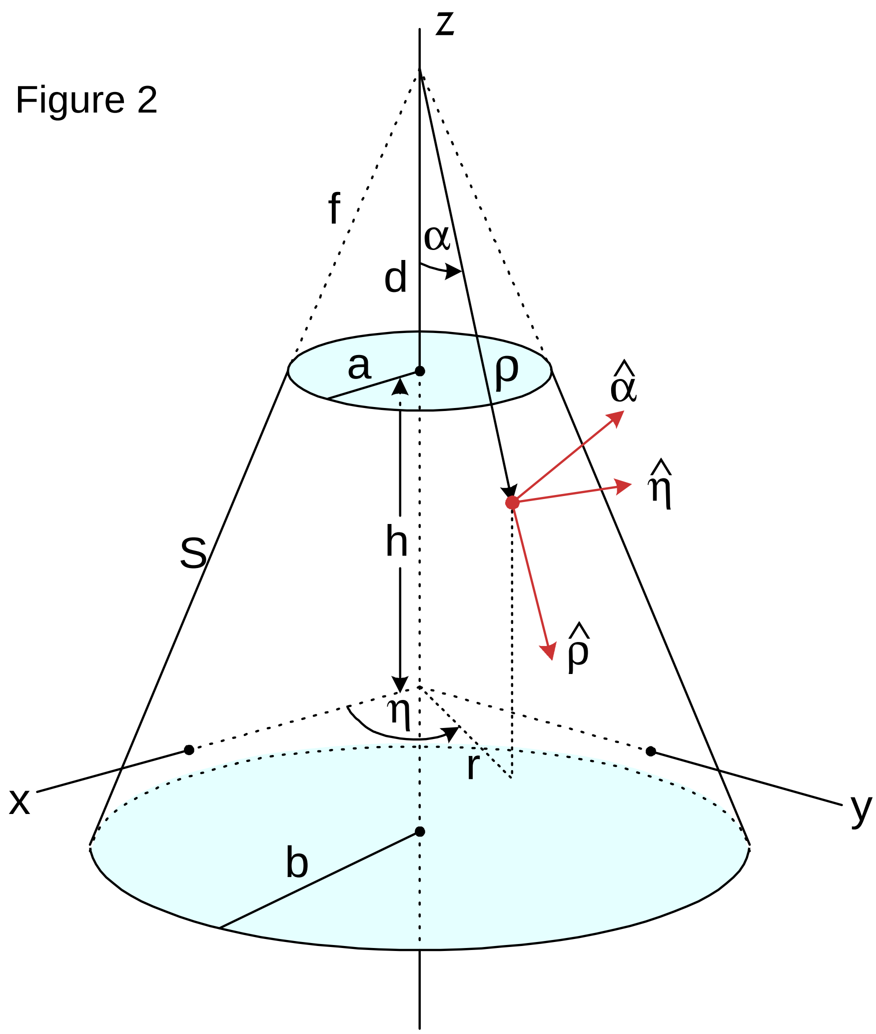


2. Torque Calculations (Cone & Top)

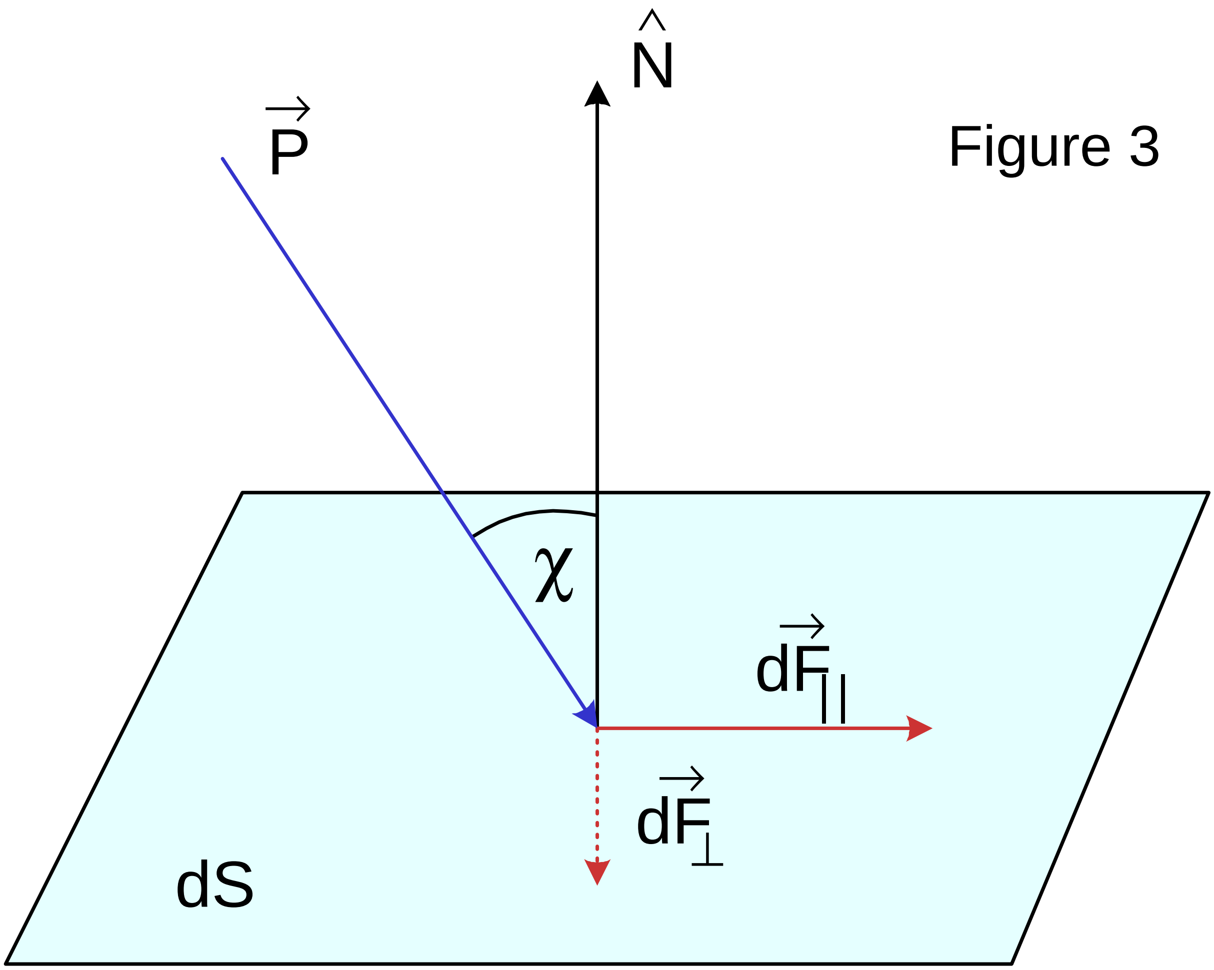


Conical Coordinate Frame

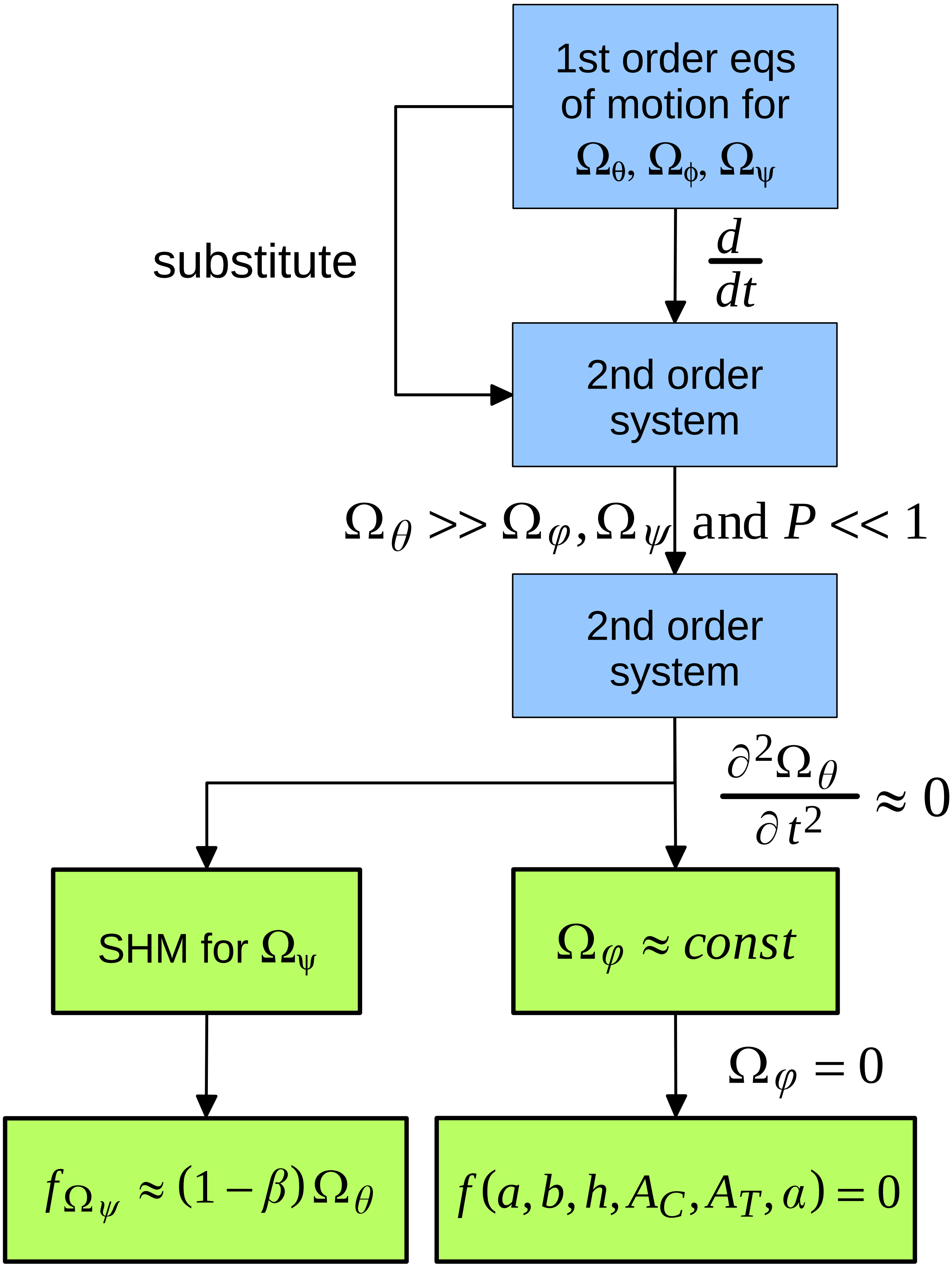
Figure 2



Force Components on Surface Element



3. Precession Calculation



Equations of Motion

- Start with the Euler equations for a rigid body:

$$\left.\begin{aligned} I_x \frac{d}{dt}\Omega_x + (I_z - I_y)\,\Omega_y\Omega_z - K_x = 0 \\ I_y \frac{d}{dt}\Omega_y + (I_x - I_z)\,\Omega_x\Omega_z - K_y = 0 \\ I_z \frac{d}{dt}\Omega_z + (I_y - I_x)\,\Omega_x\Omega_y - K_z = 0 \end{aligned}\right\}$$

- Write the components of Ω along the rotation axes as projections onto the body xyz frame:

$$\vec{\Omega}_\varphi + \vec{\Omega}_\psi + \vec{\Omega}_\theta = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} \frac{d\varphi}{dt}\sin\theta\,\sin\psi \;+\; \frac{d\psi}{dt}\cos\theta \\ \frac{d\varphi}{dt}\cos\theta\,\sin\psi \;-\; \frac{d\psi}{dt}\sin\theta \\ \frac{d\varphi}{dt}\cos\psi \;+\; \frac{d\theta}{dt} \end{bmatrix}$$

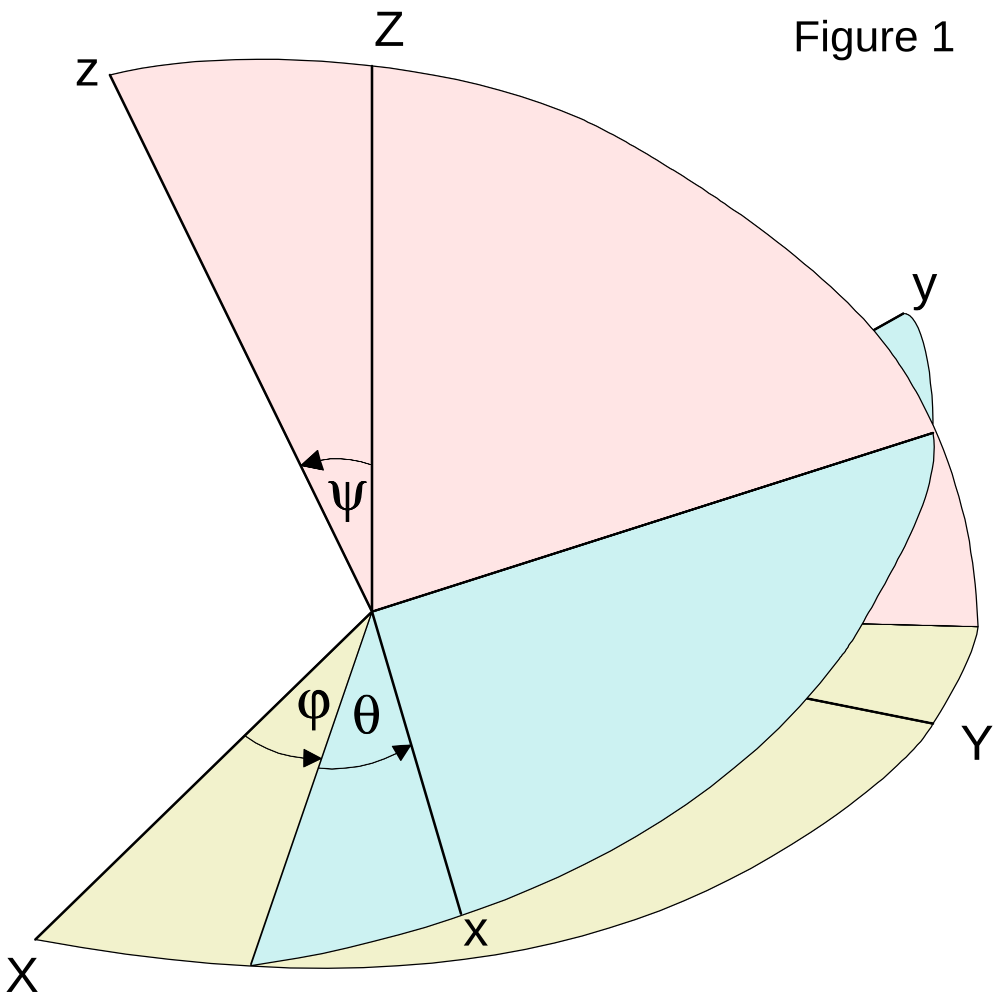
- Substitute into the Euler equations to get

$$\frac{d^2\psi}{dt^2}\cos\theta + \frac{d^2\varphi}{dt^2}\sin\theta\,\sin\psi + \left[\frac{I_z-I_y}{I_x}\left(\frac{d\varphi}{dt}\right)^2\cos\psi + \frac{I_x-I_y+I_z}{I_x}\frac{d\theta}{dt}\frac{d\varphi}{dt}\right]\sin\psi\,\cos\theta$$
$$+\left(\frac{I_x+I_y-I_z}{I_x}\frac{d\varphi}{dt}\cos\psi - \frac{I_x-I_y+I_z}{I_x}\frac{d\theta}{dt}\right)\frac{d\psi}{dt}\sin\theta - \frac{K_x}{I_x} = 0$$

$$-\frac{d^2\psi}{dt^2}\sin\theta + \frac{d^2\varphi}{dt^2}\cos\theta\,\sin\psi + \left[\frac{I_x-I_z}{I_y}\left(\frac{d\varphi}{dt}\right)^2\cos\psi + \frac{I_x-I_y-I_z}{I_y}\frac{d\theta}{dt}\frac{d\varphi}{dt}\right]\sin\psi\,\sin\theta$$
$$+\left(\frac{I_x+I_y-I_z}{I_y}\frac{d\varphi}{dt}\cos\psi + \frac{I_x-I_y-I_z}{I_y}\frac{d\theta}{dt}\right)\frac{d\psi}{dt}\cos\theta - \frac{K_y}{I_y} = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{d^2\varphi}{dt^2}\cos\psi - \frac{I_x-I_y}{I_z}\left(\frac{d\varphi}{dt}\right)^2\cos\theta\,\sin\theta\;\sin^2\psi$$
$$+\left(2\frac{I_x-I_y}{I_z}\sin^2\theta - \frac{I_x-I_y+I_z}{I_z}\right)\frac{d\psi}{dt}\frac{d\varphi}{dt}\sin\psi + \frac{I_x-I_y}{I_z}\left(\frac{d\psi}{dt}\right)^2\cos\theta\,\sin\theta - \frac{K_z}{I_z} = 0$$

Euler Angles



Symmetric Top

- We have a "symmetric top", so let $I_x = I_y = I_{xy}$ and

$$\beta \equiv \frac{I_{xy} - I_z}{I_{xy}}$$

  We find

$$\frac{d^2\psi}{dt^2}\cos\theta + \frac{d^2\varphi}{dt^2}\sin\theta\sin\psi + \left[(1-\beta)\frac{d\theta}{dt}\frac{d\varphi}{dt} - \beta\left(\frac{d\varphi}{dt}\right)^2\cos\psi\right]\sin\psi\cos\theta + \left[(1+\beta)\frac{d\varphi}{dt}\cos\psi - (1-\beta)\frac{d\theta}{dt}\right]\frac{d\psi}{dt}\sin\theta - \frac{K_x}{I_{xy}} = 0$$

$$-\frac{d^2\psi}{dt^2}\sin\theta + \frac{d^2\varphi}{dt^2}\cos\theta\sin\psi - \left[(1-\beta)\frac{d\theta}{dt}\frac{d\varphi}{dt} - \beta\left(\frac{d\varphi}{dt}\right)^2\cos\psi\right]\sin\psi\sin\theta + \left[(1+\beta)\frac{d\varphi}{dt}\cos\psi - (1-\beta)\frac{d\theta}{dt}\right]\frac{d\psi}{dt}\cos\theta - \frac{K_y}{I_{xy}} = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{d^2\varphi}{dt^2}\cos\psi - \frac{d\varphi}{dt}\frac{d\psi}{dt}\sin\psi - \frac{K_z}{(1-\beta)I_{xy}} = 0$$

- The third equation is conservation of angular momentum along the symmetry axis:

$$\frac{d}{dt}\left(\frac{d\theta}{dt} + \frac{d\varphi}{dt}\cos\psi\right) = \frac{K_z}{(1-\beta)I_{xy}}$$

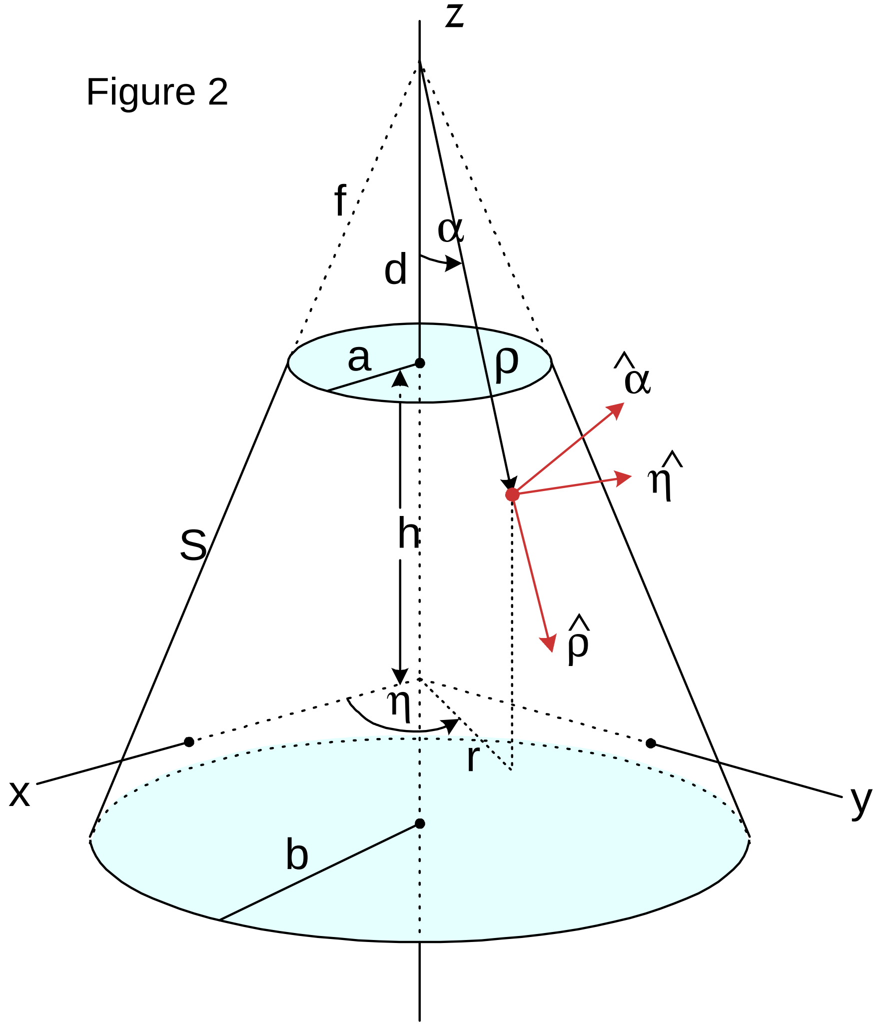
Symmetric Top (cont.)

- Convert to a system of first-order ODEs:

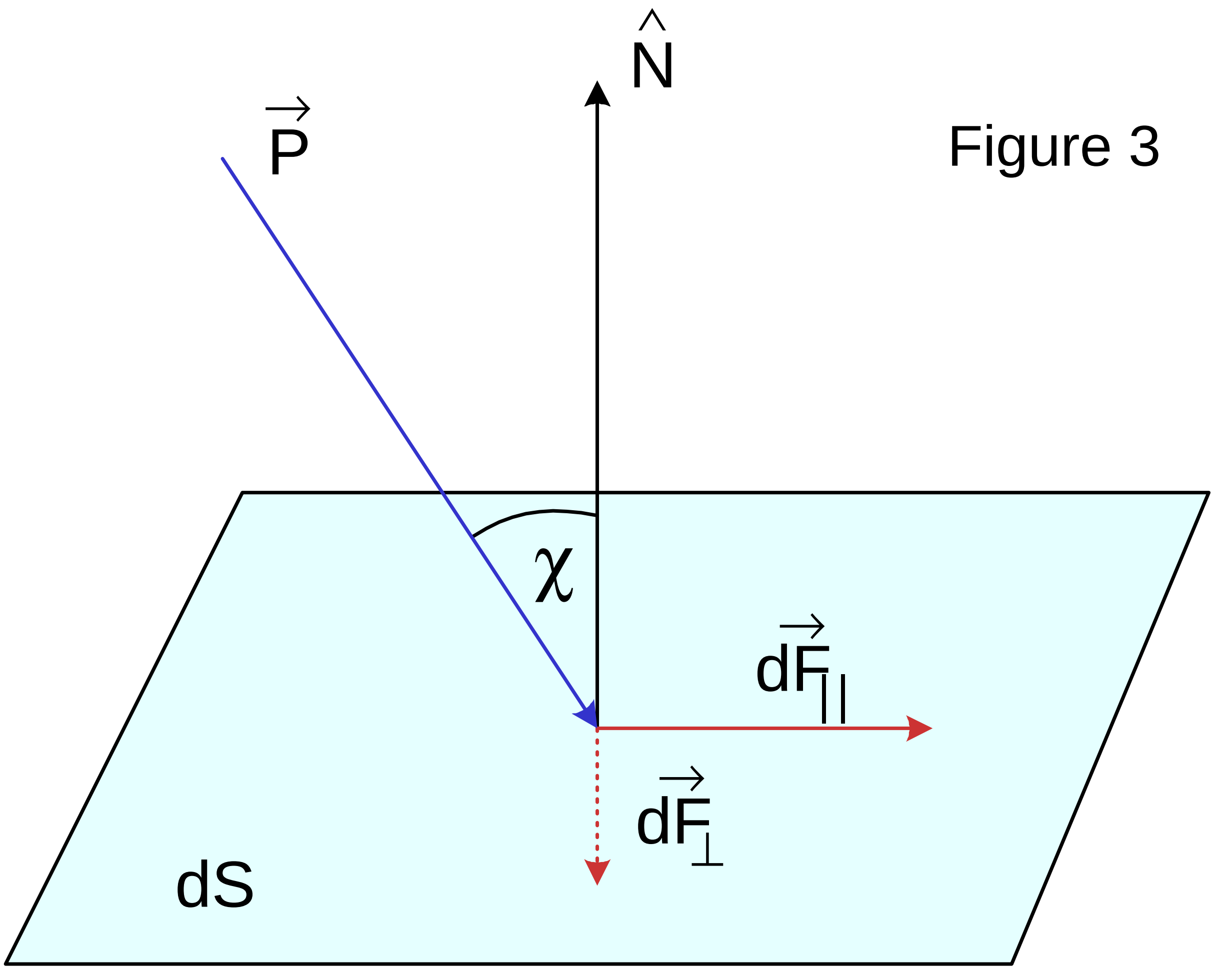
$$
\begin{aligned}
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\frac{d\psi}{dt} &= \Omega_\psi \\
\frac{d\theta}{dt} &= \Omega_\theta \\
\sin\psi \frac{d}{dt}\Omega_\varphi &= [(1-\beta)\Omega_\theta - (1+\beta)\cos\psi\,\Omega_\varphi]\Omega_\psi + \frac{K_x \sin\theta + K_y \cos\theta}{I_{xy}} \\
\frac{d}{dt}\Omega_\psi &= \left[\beta\cos\psi\,\Omega_\varphi^2 - (1-\beta)\Omega_\theta\Omega_\varphi\right]\sin\psi + \frac{K_x\cos\theta - K_y\sin\theta}{I_{xy}} \\
\sin\psi\frac{d}{dt}\Omega_\theta &= [(1+\beta\cos^2\psi)\Omega_\varphi - (1-\beta)\cos\psi\;\Omega_\theta]\Omega_\psi \\
&\quad + \frac{K_z\sin\psi}{(1-\beta)I_{xy}} - \frac{K_x\sin\theta + K_y\cos\theta}{I_{xy}}\cos\psi
\end{aligned}
$$

Conical Coordinate Frame

Figure 2



Force Components on Surface Element



Force Due to Radiation Pressure

- "Conical" coordinates:

$$\begin{aligned} x &= \rho \sin\alpha \cos\eta \\ y &= \rho \sin\alpha \sin\eta \\ z &= h + \frac{a}{\tan\alpha} - \rho\cos\alpha \end{aligned}$$

- Infinitesimal force components perpendicular to and tangent to the cone surface:

$$\begin{bmatrix} dF_\perp \\ dF_\parallel \end{bmatrix} = P \cdot d\Sigma \cdot |\cos\gamma| \cdot \begin{bmatrix} (1+A)\cos\gamma \\ (1-A)\sin\gamma \end{bmatrix}$$

- A short calculation reveals that

$$\begin{bmatrix} dF_\rho \\ dF_\eta \\ dF_\alpha \end{bmatrix} = P \cdot d\Sigma \cdot \cos\chi \cdot \begin{bmatrix} (1-A)\,\pi_\rho \\ (1-A)\,\pi_\eta \\ (1-A)\pi_\alpha - 2A\cos\chi \end{bmatrix}$$

where

$$\cos\chi \equiv -(\hat{P}\cdot\hat{N}) = -\cos\gamma$$

$$\pi_\rho \equiv \frac{P_\rho}{P} \qquad \pi_\eta \equiv \frac{P_\eta}{P} \qquad \pi_\alpha \equiv \frac{P_\alpha}{P}$$

Torque Due to Radiation Pressure

- From $$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_a \end{bmatrix} = \mathbb{R}(a,\eta)^{-1}\,\mathbb{R}(\varphi,\psi,\theta)\begin{bmatrix} \pi_X \\ \pi_Y \\ \pi_Z \end{bmatrix}$$

  we have (third component)

$$\begin{aligned}\cos\chi = -\pi_a = \; & -\{\cos a[\cos\eta(\cos\theta\cos\varphi - \sin\theta\cos\psi\sin\varphi) \\ & \qquad - \sin\eta(\sin\theta\cos\varphi + \cos\theta\cos\psi\sin\varphi)] + \sin a\sin\psi\sin\varphi\}\,\pi_X \\ & -\{\cos a[\cos\eta(\cos\theta\sin\varphi + \sin\theta\cos\psi\cos\varphi) \\ & \qquad - \sin\eta(\sin\theta\sin\varphi - \cos\theta\cos\psi\cos\varphi)] - \sin a\sin\psi\cos\varphi\}\,\pi_Y \\ & -[\cos a(\cos\eta\sin\theta\sin\psi + \sin\eta\cos\theta\sin\psi) + \sin a\cos\psi]\,\pi_Z\end{aligned}$$

$$\pi_X \equiv \frac{P_X}{P} \qquad \pi_Y \equiv \frac{P_Y}{P} \qquad \pi_Z \equiv \frac{P_Z}{P}$$

- We can integrate over the conical surface:

$$\begin{bmatrix} F_\rho \\ F_\eta \\ F_a \end{bmatrix} = P\int_0^{2\pi}\int_f^{f+S} \cos\chi \cdot \begin{bmatrix} (1-A_C)\,\pi_\rho \\ (1-A_C)\,\pi_\eta \\ (1-A_C)\,\pi_a - 2A_C\cos\chi \end{bmatrix} \cdot \rho\sin a\, d\rho\, d\eta$$

- The torque is then

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_a \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \vec{r} \times \begin{bmatrix} (1-A_C)\,\pi_\rho \\ (1-A_C)\,\pi_\eta \\ (1-A_C)\,\pi_a - 2A_C\cos\chi \end{bmatrix} \cdot \cos\chi \cdot \rho\sin a\, d\rho\, d\eta$$

Torque Due to Radiation Pressure (cont)

- The radius vector is, in conical coordinates,

$$\vec{r} = \mathbb{R}(\alpha, \eta)^{-1} \begin{bmatrix} \rho \sin\alpha \cos\eta \\ \rho \sin\alpha \sin\eta \\ h + \dfrac{a}{\tan\alpha} - \rho \cos\alpha \end{bmatrix} = \begin{bmatrix} -h\cos\alpha - a\frac{\cos^2\alpha}{\sin\alpha} + \rho \\ 0 \\ h\sin\alpha + a\cos\alpha \end{bmatrix}$$

- Perform the first integral to get

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = P(b-a)\int_0^{2\pi} \begin{bmatrix} -B_2\pi_\eta \\ B_2\pi_\rho + B_1 Q\pi_\alpha \cos\chi - 2B_1 A_C \cos\chi \\ B_2(1-A_C)P_\eta \cos\chi \end{bmatrix} \cos\chi \, d\eta$$

where

$$B_1 \equiv \tfrac{1}{2}\tfrac{1}{\sin^2\alpha}\left[U(a+b)\cos\alpha - \tfrac{2}{3}(a^2 + ab + b^2)\right]$$

$$B_2 \equiv \tfrac{1}{2}\tfrac{1}{\sin\alpha} QU(a+b)$$

$$Q \equiv 1 - A_C \qquad U \equiv h\sin\alpha + a\cos\alpha$$

Torque Due to Radiation Pressure (cont)

- Transform back to the cartesian body frame:

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = P(b-a)\int_0^{2\pi} \mathbb{R}(a,\eta) \begin{bmatrix} -B_2\pi_\eta \\ B_2\pi_\rho + B_1 Q\,\pi_\alpha \cos\chi - 2B_1 A_C \cos\chi \\ B_2(1-A_C)P_\eta \cos\chi \end{bmatrix} \cos\chi \, d\eta$$

- Perform the integral to get, finally, the total torque acting on the conical surface:

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = V \begin{bmatrix} \pi_X(\cos\psi\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \quad + \pi_Y(\sin\theta\sin\phi - \cos\psi\cos\theta\cos\phi) - \pi_Z\cos\theta\sin\psi \\ \pi_X(\cos\theta\cos\phi - \cos\psi\sin\theta\sin\phi) \\ \quad + \pi_Y(\cos\theta\sin\phi + \cos\psi\sin\theta\cos\phi) + \pi_Z\sin\theta\sin\psi \\ 0 \end{bmatrix}$$

where

$$\begin{aligned} V \equiv\ & \pi P(b-a)(-\pi_X\sin\psi\sin\phi + \pi_Y\sin\psi\cos\phi - \pi_Z\cos\psi) \\ & \cdot[B_1(3+A_C)\cos\alpha\,\sin\alpha - B_2(2\sin^2\alpha - \cos^2\alpha)] \end{aligned}$$

Torque Due to Radiation Pressure (cont)

- The contribution to the "top" surface is just a special case of the cone formulation. Hence, we find

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = W \begin{bmatrix} \pi_X(\cos\psi\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \quad + \pi_Y(\sin\theta\sin\phi - \cos\psi\cos\theta\cos\phi) - \pi_Z\cos\theta\sin\psi \\ \pi_X(\cos\theta\cos\phi - \cos\psi\sin\theta\sin\phi) \\ \quad + \pi_Y(\cos\theta\sin\phi + \cos\psi\sin\theta\cos\phi) + \pi_Z\sin\theta\sin\psi \\ 0 \end{bmatrix}$$

where

$$W \equiv \pi P a^2 h(1 - A_T)(-\pi_X \sin\psi \sin\varphi + \pi_Y \sin\psi \cos\varphi - \pi_Z \cos\psi)$$

The Equations of Motion

- Therefore, the equations of motion become

$$
\begin{aligned}
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\frac{d\psi}{dt} &= \Omega_\psi \\
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\sin\psi\frac{d}{dt}\Omega_\varphi &= [(1-\beta)\,\Omega_\theta - (1+\beta)\cos\psi\,\Omega_\varphi]\,\Omega_\psi + K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\
\frac{d}{dt}\Omega_\psi &= \left[\beta\cos\psi\,\Omega_\varphi^2 - (1-\beta)\,\Omega_\theta\,\Omega_\varphi\right]\sin\psi + K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\
\sin\psi\frac{d}{dt}\Omega_\theta &= [(1+\beta\cos^2\psi)\,\Omega_\varphi - (1-\beta)\cos\psi\,\Omega_\theta]\,\Omega_\psi + K_3(a,b,h,\alpha,A_C,A_T,\varphi,\psi)
\end{aligned}
$$

$$
\begin{aligned}
K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &\equiv G(a,b,h,\alpha,A_C,A_T)\cdot g_1(\varphi,\psi) \\
K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &\equiv G(a,b,h,\alpha,A_C,A_T)\cdot g_2(\varphi,\psi) \\
K_3(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &\equiv G(a,b,h,\alpha,A_C,A_T)\cdot g_3(\varphi,\psi)
\end{aligned}
$$

$$
\begin{aligned}
g_0(\varphi,\psi) &= -\pi_X\sin\varphi\,\sin\psi + \pi_Y\sin\psi\,\cos\varphi - \pi_Z\cos\psi \\
g_1(\varphi,\psi) &= \quad g_0(\varphi,\psi)\cdot(\pi_X\cos\varphi + \pi_Y\sin\varphi) \\
g_2(\varphi,\psi) &= \quad g_0(\varphi,\psi)\cdot(\pi_X\cos\psi\,\sin\varphi - \pi_Y\cos\psi\,\cos\varphi - \pi_Z\sin\psi) \\
g_3(\varphi,\psi) &= -g_1(\varphi,\psi)\cos\psi
\end{aligned}
$$

$$
\begin{aligned}
G(a,b,h,\alpha,A_C,A_T) &= G_C(a,b,h,\alpha,A_C) + G_T(a,h,A_T) \\
G_C(a,b,h,\alpha,A_C) &= \frac{\pi P}{I_{xy}}(b-a)\Big[(1-A_C+2A_C\cos^2\alpha)(h\sin\alpha + a\cos\alpha)\frac{a+b}{\sin\alpha} \\
&\qquad - \tfrac{1}{3}(3+A_C)\cos\alpha\,\frac{a^2+a\,b+b^2}{\sin\alpha}\Big] \\
G_T(a,h,A_T) &= \frac{\pi P}{I_{xy}}(1-A_T)\,a^2h
\end{aligned}
$$

Fast Spin Approximation — Precession

- Assume a fast-spinning craft. Then $\Omega_\theta >> \Omega_\varphi, \Omega_\psi$
  Differentiate, substitute, and perform a series expansion on Ω_ψ, Ω_φ, and *P* to get

$$
\begin{aligned}
\sin\psi \frac{d^2}{dt^2}\Omega_\varphi &= -(1-\beta)^2\Omega_\theta^2\Omega_\varphi \sin\psi + (1-\beta)\Omega_\theta K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\
\frac{d^2}{dt^2}\Omega_\psi &= -(1-\beta)^2\Omega_\theta^2\Omega_\psi - (1-\beta)\Omega_\theta K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\
\sin\psi \frac{d^2}{dt^2}\Omega_\theta &= [(1-\beta)^2\Omega_\theta^2\Omega_\varphi \sin\psi - (1-\beta)\Omega_\theta K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)]\cos\psi
\end{aligned}
$$

- From the first or third equations, we have

$$\Omega_\varphi \approx \frac{K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)}{(1-\beta)\Omega_\theta \sin\psi}$$

- For a flat disk of uniform albedo A, this reduces to

$$\Omega_\varphi \approx \frac{(1-A)\pi b^2 h}{(1-\beta) I_{xy}\Omega_\theta} P\cos\psi$$

  which agrees with your average #10 business envelope.

Fast Spin Approximation — Nutation

- From the second equation we have simple harmonic motion for Ω_ψ:

$$\Omega_\psi \approx A\cos[(1-\beta)\,\Omega_\theta\, t] + B\sin[(1-\beta)\Omega_\theta\, t] - \frac{K_1(a,b,h,\alpha\,,A_C,A_T,\varphi,\psi)}{(1-\beta)\Omega_\theta}$$

- The third term is constant for $\pi_X, \pi_Y \to 0,\ \pi_Z \to 1$. This implies a small, monotonic drift in the inclination angle ψ. However, it is actually the first term in the expansion of a large-period oscillation.

Cone Angle for Precession Nulling

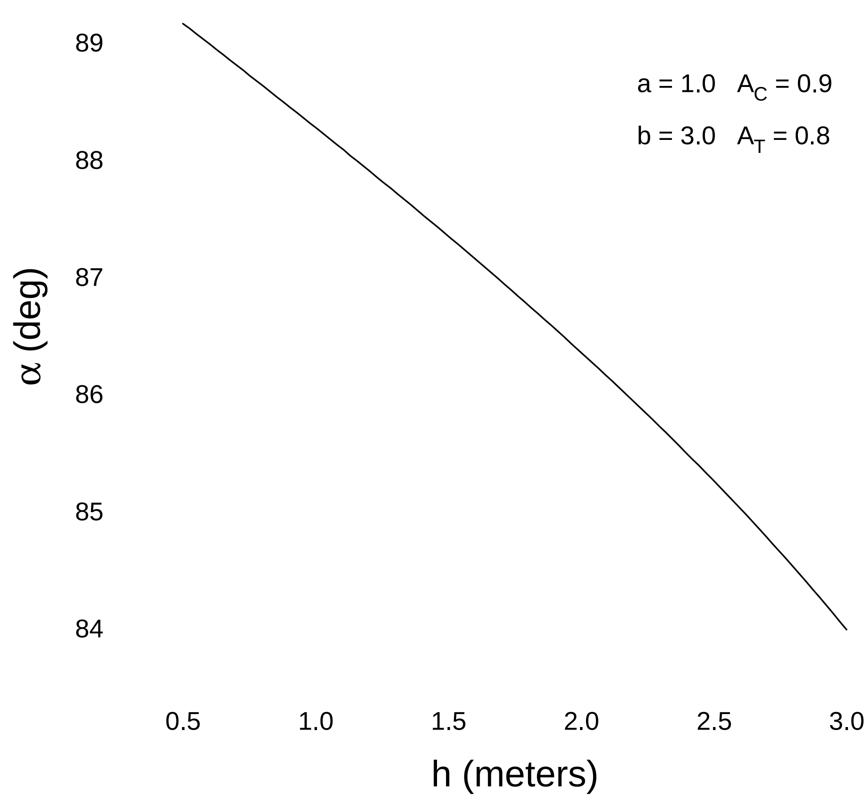
- Precession null:

$$G(a, b, h, \alpha, A_C, A_T) = 0$$

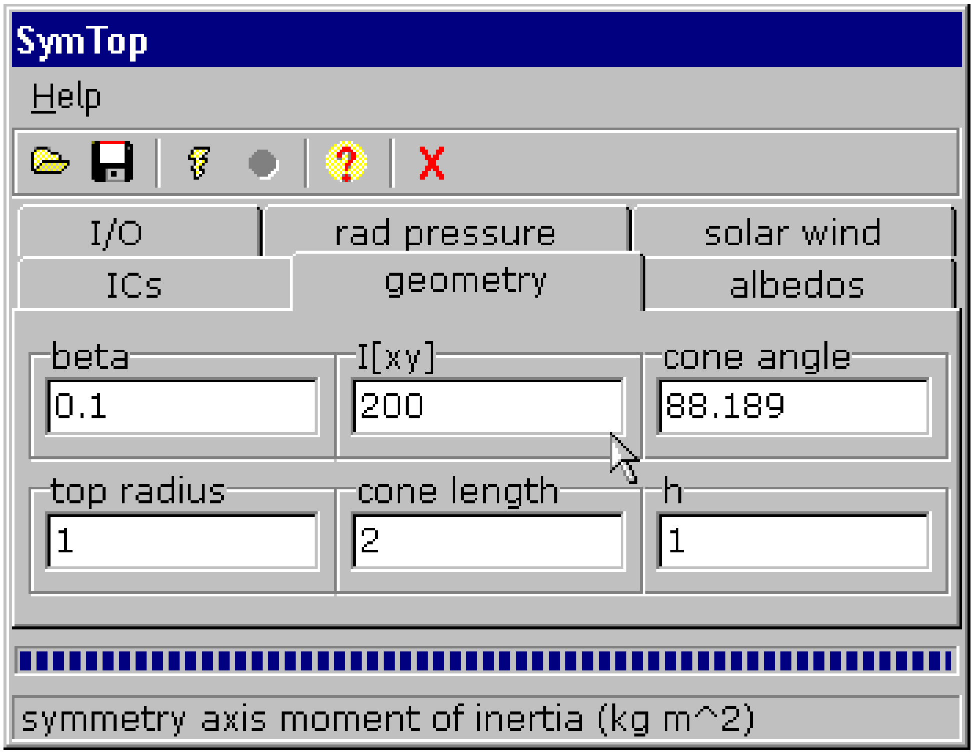
- This is equivalent to

$$(b-a)[(1-A_C+2A_C\cos^2\alpha)(h\sin\alpha+a\cos\alpha)(a+b) - \tfrac{1}{3}(3+A_C)(a^2+ab+b^2)\cos\alpha\,] + (1-A_T)\,a^2 h\sin\alpha \;=\; 0$$

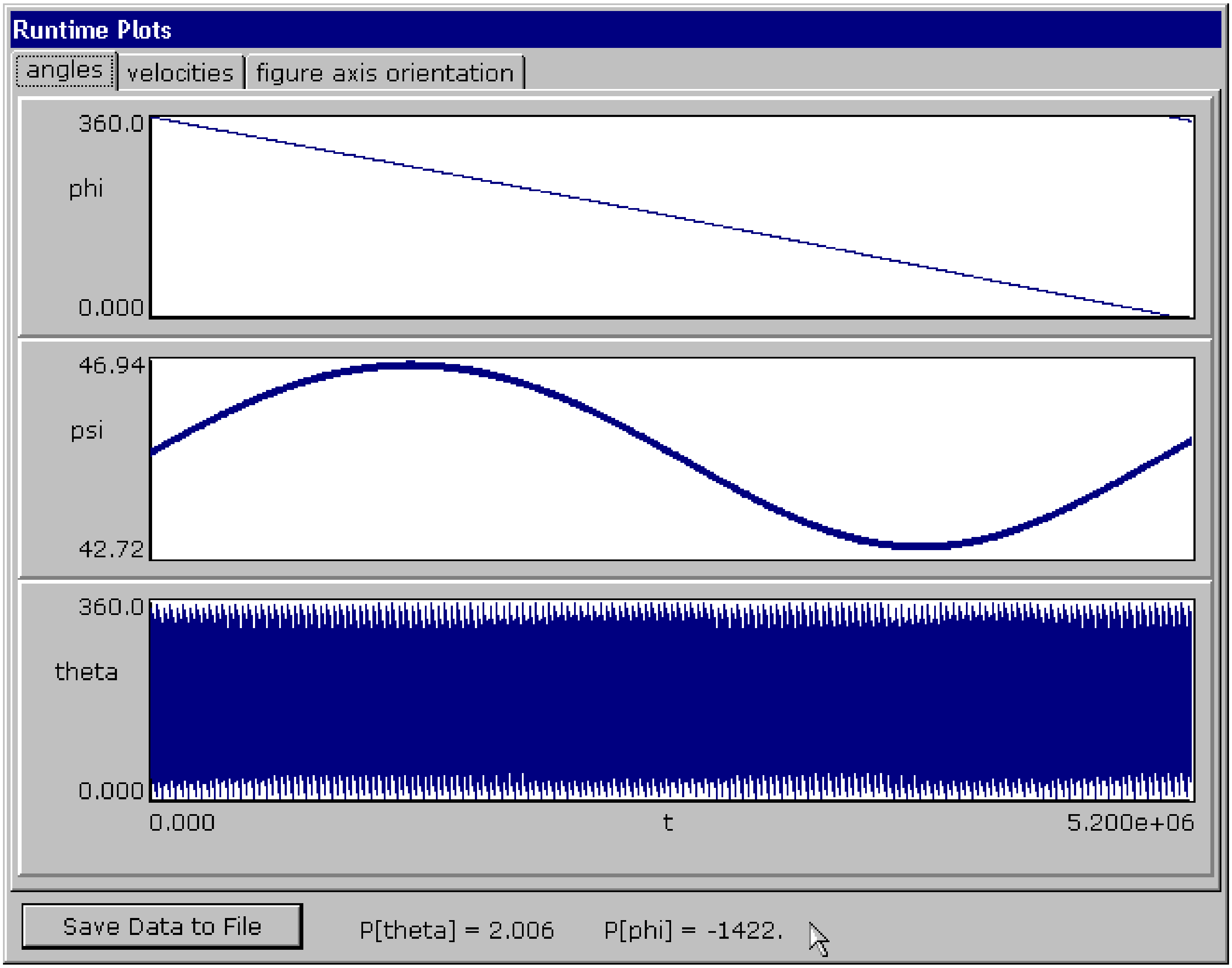
Nulling Cone Angle vs. h

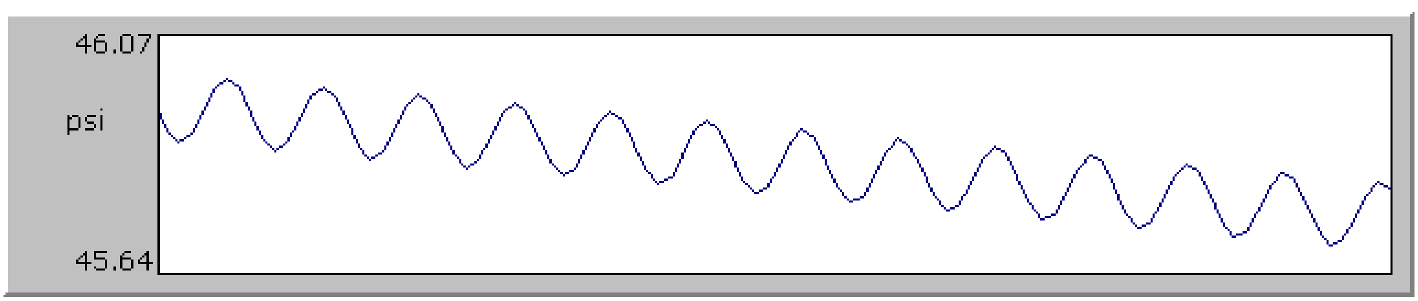


SymTop


SymTop
Help
I/O
rad pressure
solar wind
ICs
geometry
albedos
beta
0.1
I[xy]
200
cone angle
88.189
top radius
1
cone length
2
h
1
symmetry axis moment of inertia (kg m^2)

FAME 60-Day Precession





Precession Period vs. (α,A_C)

- Shield angle will need to be adjustable in flight, at least at beginning of mission.