Characterization of Approximate Solutions of Kepler's Equation and Optimization of Time Series Storage

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Introduction

Kepler's equation can be written $f(\phi) = 0$, where $f(\phi) = \phi - e \sin(\phi) - M$, and where ϕ is the eccentric anomaly, $M = n\,t$ is the mean anomaly, n is the mean motion, and e is the orbital eccentricity. Let x be an approximation for the true solution ϕ, and let δ be the error in the guess: $\phi = \xi + \delta$. Then we can expand $f(\xi + \delta)$ in a Taylor series. For example, to third order we have

restart

with(*DEtools*)

$f(\phi) = \phi - e\sin(\phi) - M$

Kepler := fn(rhs(%), ϕ, *e*, *M*)

$$Kepler := (\phi, e, M) \rightarrow \phi - e\sin(\phi) - M$$

$f(\phi) = \text{convert}(\text{expansion}(f(\xi + \delta), \delta, 3), diff)$

$$f(\phi) = f(\xi) + \left(\frac{\partial}{\partial\xi} f(\xi)\right)\delta + \frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2} f(\xi)\right)\delta^2 + \frac{1}{6}\left(\frac{\partial^3}{\partial\xi^3} f(\xi)\right)\delta^3$$

ODE := %

This is equivalent to the differential operator form

$f(\phi) = \text{de2diffop}(\text{rhs}(ODE), f(\xi), [Dx, \xi])\, f(\xi)$

$$f(\phi) = \text{de2diffop}\left(f(\xi) + \left(\frac{\partial}{\partial\xi} f(\xi)\right)\delta + \frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2} f(\xi)\right)\delta^2 + \frac{1}{6}\left(\frac{\partial^3}{\partial\xi^3} f(\xi)\right)\delta^3, f(\xi), [Dx, \xi]\right) f(\xi)$$

where the argument in parenthesis is a differential operator with $Dx\, f = \frac{\partial}{\partial\xi} f$.

Expansion in eccentricity

Take the old-fashioned approach and expand the solution in a polynomial in eccentricity.

trialsoln := **proc**(ϕ, *e*, *C*, *n*::{*posint*, *name*}) **local** *k*; ϕ = sum(*C*[*k*]**e*^*k*, *k* = 0 .. *n*) **end**

Ne := 6

trialsoln(ϕ, *e*, *C*, *Ne*)

trial := %

$$\phi = C_0 + C_1 e + C_2 e^2 + C_3 e^3 + C_4 e^4 + C_5 e^5 + C_6 e^6$$

Stick this back into the Kepler equation, then expand on eccentricity to get

subs(*trial*, Kepler(ϕ, *e*, *M*) = 0)

expansion(%, *e*, *Ne*)

eqnx := %

$$\left(-\sin(C_0)\left(\frac{1}{6} C_2 C_1^3 - C_4 C_1 - C_3 C_2\right) - \cos(C_0)\left(\frac{1}{120} C_1^5 - \frac{1}{2} C_2^2 C_1 + C_5 - \frac{1}{2} C_3 C_1^2\right) + C_6\right) e^6 + \left(C_5 - \cos(C_0)\left(-\frac{1}{2} C_2 C_1^2 + C_4\right) - \sin(C_0)\left(-C_3 C_1 + \frac{1}{24} C_1^4 - \frac{1}{2} C_2^2\right)\right) e^5 + \left(C_4 + \sin(C_0) C_2 C_1 - \cos(C_0)\left(C_3 - \frac{1}{6} C_1^3\right)\right) e^4 + \left(\frac{1}{2}\sin(C_0) C_1^2 - \cos(C_0) C_2 + C_3\right) e^3$$

$$+(-\cos(C_0)\,C_1 + C_2)\,e^2 + (C_1 - \sin(C_0))\,e + C_0 - M = 0$$

Solve successively for the coefficients. We'll write a quickie procedure to do this automatically.

```
xsolve := proc(expr::algebraic, var::name, coef::name)
local k, kmax, eqn, sols;
    kmax := degree(collect(expr, var), var);
    sols := [ ];
    for k from 0 to kmax do
        eqn := coeff(expr, var, k); sols := [op(sols), isolate(subs(sols, eqn), coef[k])]
    od;
    sols
end
```

```
xsolve(lhs(eqnx), e, C)
```

```
trial_coeffs := %
```

$$\Big[C_0 = M,\ C_1 = \sin(M),\ C_2 = \cos(M)\sin(M),\ C_3 = -\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M),$$

$$C_4 = -\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right),\ C_5 =$$

$$\cos(M)\left(-\frac{3}{2}\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)$$

$$+\sin(M)\left(-\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\sin(M) + \frac{1}{24}\sin(M)^4 - \frac{1}{2}\cos(M)^2\sin(M)^2\right),\ C_6 =$$

$$\sin(M)\Bigg(\frac{1}{6}\cos(M)\sin(M)^4$$

$$-\left(-\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)\sin(M)$$

$$-\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\cos(M)\sin(M)\Bigg) + \cos(M)\Bigg(\frac{1}{120}\sin(M)^5 - \frac{1}{2}\cos(M)^2\sin(M)^3$$

$$+\cos(M)\left(-\frac{3}{2}\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)$$

$$+\sin(M)\left(-\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\sin(M) + \frac{1}{24}\sin(M)^4 - \frac{1}{2}\cos(M)^2\sin(M)^2\right)$$

$$-\frac{1}{2}\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\sin(M)^2\Bigg)\Big]$$

Check:

```
subs(%, eqnx)
```

$$0 = 0$$

Hence, the approximate solution is

subs(*trial_coeffs*, *trial*)

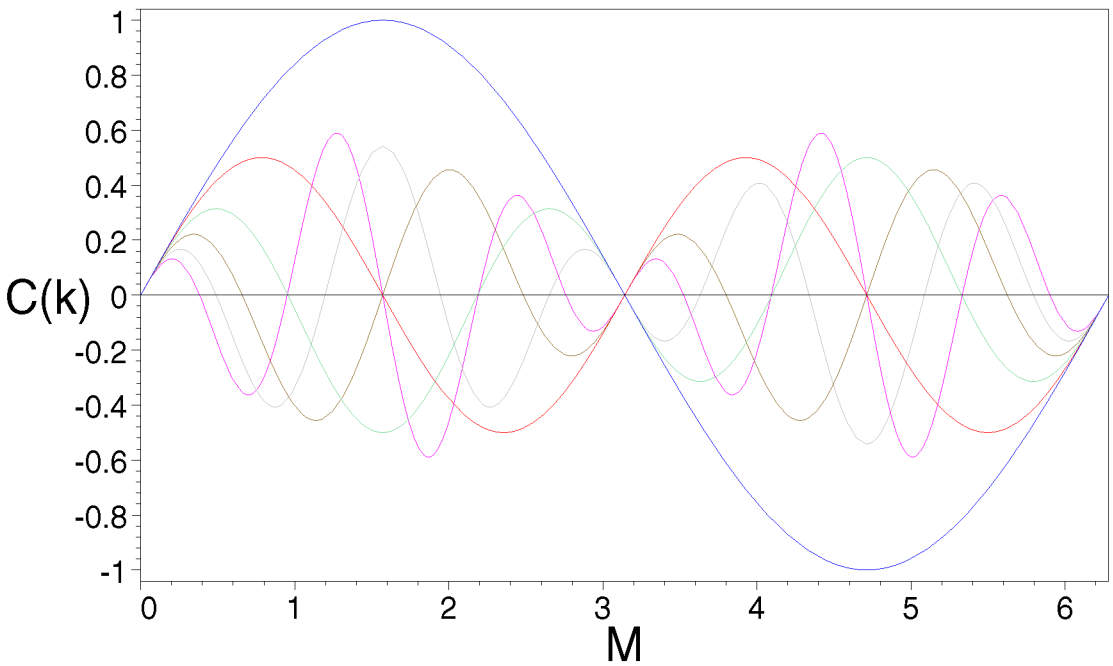
soln := %

$$\phi = M + \sin(M)\, e + \cos(M)\sin(M)\, e^2 + \left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)e^3$$
$$+\left(-\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)e^4 + \Bigg($$
$$\cos(M)\left(-\frac{3}{2}\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)$$
$$+\sin(M)\left(-\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\sin(M) + \frac{1}{24}\sin(M)^4 - \frac{1}{2}\cos(M)^2\sin(M)^2\right)\Bigg)e^5 + \Bigg($$
$$\sin(M)\Bigg(\frac{1}{6}\cos(M)\sin(M)^4$$
$$-\left(-\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)\sin(M)$$
$$-\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\cos(M)\sin(M)\Bigg) + \cos(M)\Bigg(\frac{1}{120}\sin(M)^5 - \frac{1}{2}\cos(M)^2\sin(M)^3$$
$$+\cos(M)\left(-\frac{3}{2}\sin(M)^3\cos(M) + \cos(M)\left(-\frac{2}{3}\sin(M)^3 + \cos(M)^2\sin(M)\right)\right)$$
$$+\sin(M)\left(-\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\sin(M) + \frac{1}{24}\sin(M)^4 - \frac{1}{2}\cos(M)^2\sin(M)^2\right)$$
$$-\frac{1}{2}\left(-\frac{1}{2}\sin(M)^3 + \cos(M)^2\sin(M)\right)\sin(M)^2\Bigg)\Bigg)e^6$$

Plot just the coefficients.

plot

plot([0, seq(rhs($trial_coeffs_k$), $k = 2 .. Ne + 1$)], $M = 0 .. 2\pi$, *color* = *mycolors*, *labels* = ["M", "C(k)"])



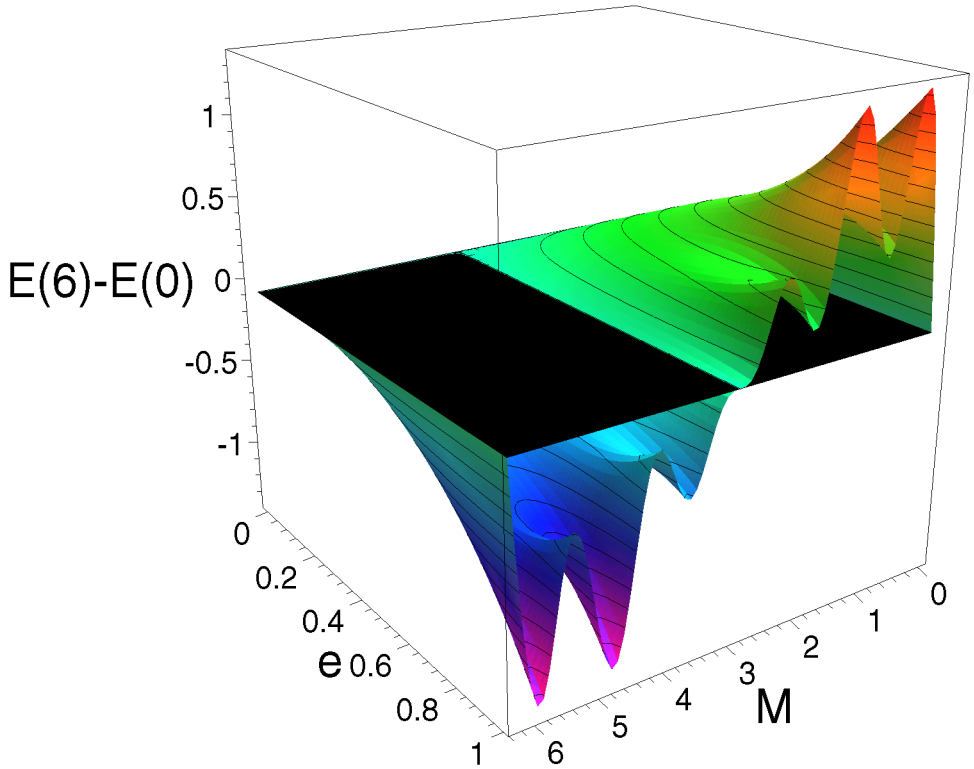
Plot the difference between the approximate solution and the mean anomaly M (which is the zeroth-order approximation), as a function of eccentricity and M.

plot

$p0 := \mathrm{plot3d}(0, M = 0 .. 2\pi, e = 0 .. 1, color = black)$

$p1 :=$
$\mathrm{plot3d}(\mathrm{rhs}(soln) - M, M = 0 .. 2\pi, e = 0 .. 1, style = patchcontour, contours = 20, grid = [75, 75])$

$plots_{display3d}([p0, p1], orientation = [55, 70],$
$labels = [\text{"M"}, \text{"e"}, \mathrm{cat}(\text{"E("}, \mathrm{convert}(Ne, string), \text{")-E(0)"})], lightmodel = light3)$



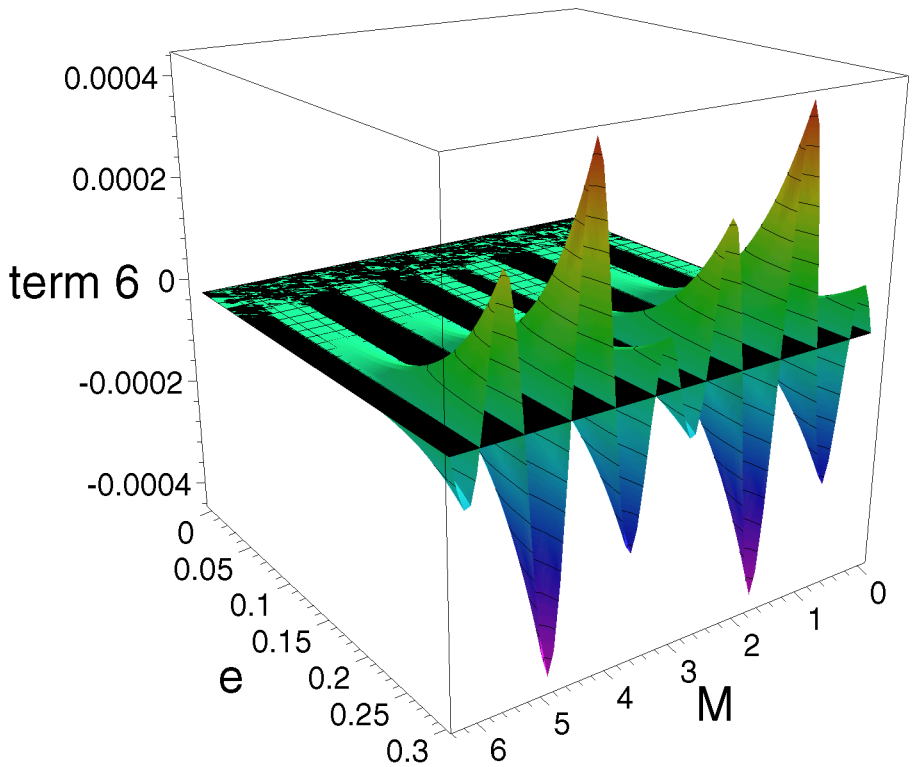
Plot only the highest-order term of the approximate solution.

plot

$p0 := \mathrm{plot3d}(0, M = 0 .. 2\,\pi, e = 0 .. .3, color = black)$

$p1 := \mathrm{plot3d}(\mathrm{rhs}(trial_coeffs_{Ne+1})\, e^{Ne}, M = 0 .. 2\,\pi, e = 0 .. .3, style = patchcontour, contours = 15, grid = [75, 75])$

$plots_{display3d}([p0, p1], orientation = [55, 70], labels = [\text{"M"}, \text{"e"}, \mathrm{cat}(\text{"term "}, \mathrm{convert}(Ne, string))])$



Create a function to numerically solve for the eccentric anomaly, then plot the error of the approximate solution.

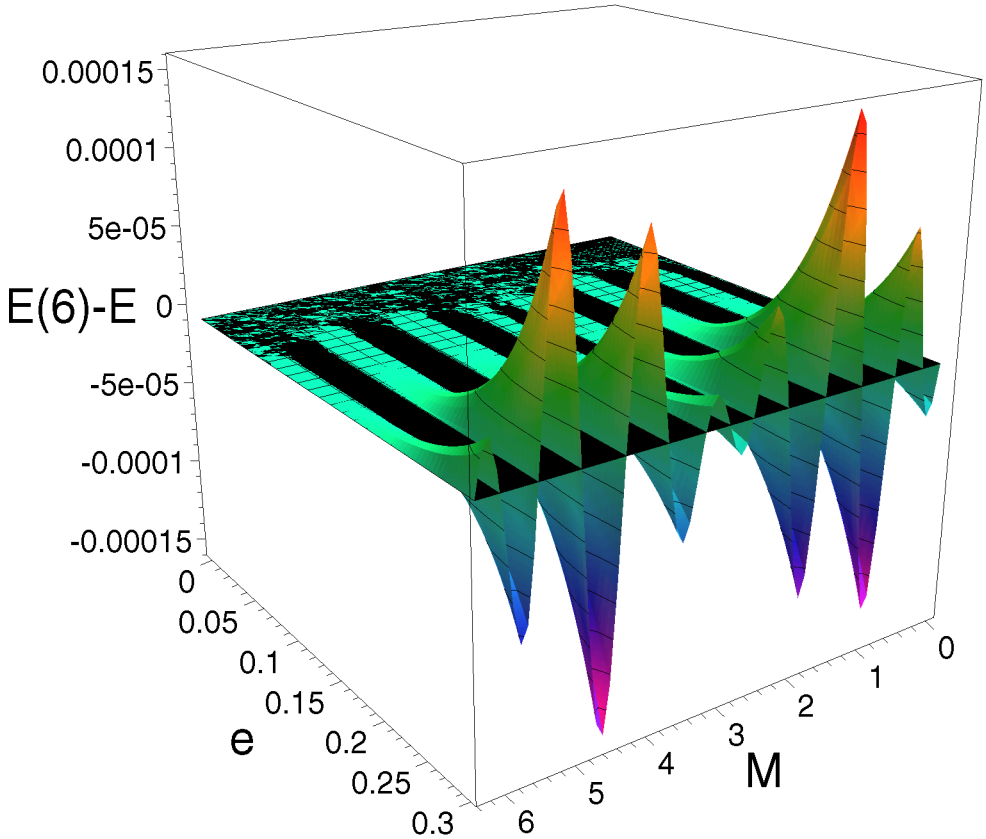
Keplersolve := **proc**(*e*, *M*) fsolve(*E* − *e*∗sin(*E*) − *M*, *E*) **end**

plot

p0 := plot3d(0, *M* = 0 .. 2 π, *e* = 0 .. .3, *color* = *black*)

p1 := plot3d(rhs(*soln*) − 'Keplersolve(*e*, *M*)', *M* = 0 .. 2 π, *e* = 0 .. .3, *style* = *patchcontour*, *contours* = 15, *grid* = [75, 75])

$plots_{display3d}$([*p0*, *p1*], *orientation* = [55, 70], *lightmodel* = *light3*, *labels* = ["M", "e", cat("E(", convert(*Ne*, *string*), ")-E")])



$Ne := 'Ne'$

Expansion in time

We would also like to characterize the solution near a particular arbitrary time, say M_0. Substitute $M = M_0 + \delta$ and expand on δ. For illustration, we look at just the second-order expansion.

long expression

expansion(subs($M = M_0 + \delta$, *soln*), δ, 2)

$$\phi = \left(-\frac{1}{2}\sin(M_0)\,e - 2\cos(M_0)\sin(M_0)\,e^2 + \left(-\frac{1}{2}\sin(M_0)\left(-\sin(M_0)^2 + \cos(M_0)^2\right) + \frac{1}{4}\sin(M_0)^3 - \frac{7}{2}\cos(M_0)^2\sin(M_0) + \left(-\cos(M_0)^2 + \sin(M_0)^2\right)\sin(M_0)\right)e^3 + \left(\frac{7}{2}\sin(M_0)^3\cos(M_0)\right.$$

$$-\left(\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)-\frac{1}{2}\sin(M_0)^3+2\cos(M_0)^2\sin(M_0)\right)\cos(M_0)+$$
$$\cos(M_0)\left(-\frac{2}{3}\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)+\frac{1}{3}\sin(M_0)^3-\frac{23}{6}\cos(M_0)^2\sin(M_0)\right.$$
$$\left.+\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)\right)-\frac{1}{2}\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)$$
$$\left.-\sin(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\right)e^4+\left(\cos(M_0)\left(\frac{21}{4}\sin(M_0)^3\cos(M_0)\right.\right.$$
$$-\frac{3}{2}\left(\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)-\frac{1}{2}\sin(M_0)^3+2\cos(M_0)^2\sin(M_0)\right)\cos(M_0)+$$
$$\cos(M_0)\left(-\frac{2}{3}\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)+\frac{1}{3}\sin(M_0)^3-\frac{23}{6}\cos(M_0)^2\sin(M_0)\right.$$
$$\left.+\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)\right)-\frac{1}{2}\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)$$
$$\left.-\sin(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\right)$$
$$\left.-\frac{1}{2}\cos(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)\right)-$$
$$\sin(M_0)\left(\frac{3}{2}\sin(M_0)^4-\frac{9}{2}\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\right.$$
$$\left.-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+\sin(M_0)\left(\right.$$
$$\frac{1}{2}\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)-\left(-\frac{1}{2}\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)\right.$$
$$\left.+\frac{1}{4}\sin(M_0)^3-\frac{7}{2}\cos(M_0)^2\sin(M_0)+\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)\right)\sin(M_0)$$
$$-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\cos(M_0)+\frac{1}{12}\sin(M_0)^2\left(-\sin(M_0)^2+\cos(M_0)^2\right)$$
$$+\frac{13}{6}\cos(M_0)^2\sin(M_0)^2-\frac{1}{2}\cos(M_0)^2\left(-\sin(M_0)^2+\cos(M_0)^2\right)$$
$$\left.-\frac{1}{2}\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)^2\right)-\frac{1}{2}\sin(M_0)$$
$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)+$$
$$\cos(M_0)\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\right.$$
$$\left.-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)+\frac{7}{6}\sin(M_0)^3\cos(M_0)-\cos(M_0)^3\sin(M_0)\right)$$

$$\Big)e^5+\Big(\sin(M_0)\Big(\frac{1}{6}\cos(M_0)\Big(2\sin(M_0)^2\Big(-\sin(M_0)^2+\cos(M_0)^2\Big)+4\cos(M_0)^2\sin(M_0)^2\Big)$$
$$-\frac{3}{4}\cos(M_0)\sin(M_0)^4$$
$$+\frac{1}{2}\Big(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\Big(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\Big)\Big)\sin(M_0)-\Big($$
$$\frac{7}{2}\sin(M_0)^3\cos(M_0)$$
$$-\Big(\sin(M_0)\Big(-\sin(M_0)^2+\cos(M_0)^2\Big)-\frac{1}{2}\sin(M_0)^3+2\cos(M_0)^2\sin(M_0)\Big)\cos(M_0)+$$
$$\cos(M_0)\Big(-\frac{2}{3}\sin(M_0)\Big(-\sin(M_0)^2+\cos(M_0)^2\Big)+\frac{1}{3}\sin(M_0)^3-\frac{23}{6}\cos(M_0)^2\sin(M_0)$$
$$+\Big(-\cos(M_0)^2+\sin(M_0)^2\Big)\sin(M_0)\Big)-\frac{1}{2}\cos(M_0)\Big(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)$$
$$-\sin(M_0)\Big(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\Big)\Big)\Big)\sin(M_0)-\Big(\sin(M_0)^4$$
$$-3\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\Big(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\Big)$$
$$-\sin(M_0)\Big(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\Big)\cos(M_0)$$
$$+\frac{1}{2}\Big(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\cos(M_0)\sin(M_0)-\Big($$
$$-\frac{1}{2}\Big(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\cos(M_0)+\Big(-\frac{1}{2}\sin(M_0)\Big(-\sin(M_0)^2+\cos(M_0)^2\Big)$$
$$+\frac{1}{4}\sin(M_0)^3-\frac{7}{2}\cos(M_0)^2\sin(M_0)+\Big(-\cos(M_0)^2+\sin(M_0)^2\Big)\sin(M_0)\Big)\cos(M_0)$$
$$-\Big(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\Big)\sin(M_0)\Big)\sin(M_0)-\Big($$
$$-\Big(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\sin(M_0)$$
$$+\Big(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\Big)\cos(M_0)\Big)\cos(M_0)\Big)-\frac{1}{2}\sin(M_0)\Big($$
$$\frac{1}{6}\cos(M_0)\sin(M_0)^4$$
$$-\Big(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\Big(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\Big)\sin(M_0)$$
$$-\Big(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\Big)\cos(M_0)\sin(M_0)\Big)+\cos(M_0)\Big(\frac{2}{3}\cos(M_0)^2\sin(M_0)^3$$

$$-\frac{1}{6}\sin(M_0)^5$$
$$-\left(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)\cos(M_0)-\Big($$
$$\sin(M_0)^4-3\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$
$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Big)\sin(M_0)$$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)^2-\Big($$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)$$
$$+\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\cos(M_0)\Big)\sin(M_0)\Big)+\cos(M_0)\Big($$
$$\frac{1}{120}\sin(M_0)\left(2\sin(M_0)^2\left(-\sin(M_0)^2+\cos(M_0)^2\right)+4\cos(M_0)^2\sin(M_0)^2\right)-\frac{1}{240}\sin(M_0)^5$$
$$+\frac{91}{30}\cos(M_0)^2\sin(M_0)^3$$
$$-\frac{1}{2}\cos(M_0)^2\left(\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)-\frac{1}{2}\sin(M_0)^3+2\cos(M_0)^2\sin(M_0)\right)$$
$$-\frac{1}{2}\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)^3+\cos(M_0)\Big(\frac{21}{4}\sin(M_0)^3\cos(M_0)$$
$$-\frac{3}{2}\left(\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)-\frac{1}{2}\sin(M_0)^3+2\cos(M_0)^2\sin(M_0)\right)\cos(M_0)+$$
$$\cos(M_0)\Big(-\frac{2}{3}\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)+\frac{1}{3}\sin(M_0)^3-\frac{23}{6}\cos(M_0)^2\sin(M_0)$$
$$+\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)\Big)-\frac{1}{2}\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)$$
$$-\sin(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\Big)$$
$$-\frac{1}{2}\cos(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)-$$
$$\sin(M_0)\Big(\frac{3}{2}\sin(M_0)^4-\frac{9}{2}\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$
$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Big)+\sin(M_0)\Big($$
$$\frac{1}{2}\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)-\Big(-\frac{1}{2}\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)$$

$$+\frac{1}{4}\sin(M_0)^3-\frac{7}{2}\cos(M_0)^2\sin(M_0)+\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)\Bigg)\sin(M_0)$$

$$-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\cos(M_0)+\frac{1}{12}\sin(M_0)^2\left(-\sin(M_0)^2+\cos(M_0)^2\right)$$

$$+\frac{13}{6}\cos(M_0)^2\sin(M_0)^2-\frac{1}{2}\cos(M_0)^2\left(-\sin(M_0)^2+\cos(M_0)^2\right)$$

$$-\frac{1}{2}\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)^2\Bigg)-\frac{1}{2}\sin(M_0)$$

$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)+$$

$$\cos(M_0)\Bigg(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)$$

$$-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)+\frac{7}{6}\sin(M_0)^3\cos(M_0)-\cos(M_0)^3\sin(M_0)\Bigg)$$

$$-\frac{1}{2}\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\left(-\sin(M_0)^2+\cos(M_0)^2\right)-\frac{1}{2}\Bigg($$

$$-\frac{1}{2}\sin(M_0)\left(-\sin(M_0)^2+\cos(M_0)^2\right)+\frac{1}{4}\sin(M_0)^3-\frac{7}{2}\cos(M_0)^2\sin(M_0)$$

$$+\left(-\cos(M_0)^2+\sin(M_0)^2\right)\sin(M_0)\Bigg)\sin(M_0)^2$$

$$-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)\cos(M_0)\Bigg)-\frac{1}{2}\cos(M_0)\Bigg(\frac{1}{120}\sin(M_0)^5$$

$$-\frac{1}{2}\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)^2-\frac{1}{2}\cos(M_0)^2\sin(M_0)^3$$

$$+\cos(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+$$

$$\sin(M_0)$$

$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)\Bigg)-$$

$$\sin(M_0)\Bigg(\frac{25}{24}\cos(M_0)\sin(M_0)^4-\frac{3}{2}\cos(M_0)^3\sin(M_0)^2+\cos(M_0)\Bigg(\frac{3}{2}\sin(M_0)^4$$

$$-\frac{9}{2}\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$

$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Bigg)$$

$$-\sin(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+$$

$$\sin(M_0)\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\right.$$
$$\left.-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)+\frac{7}{6}\sin(M_0)^3\cos(M_0)-\cos(M_0)^3\sin(M_0)\right)$$
$$+\cos(M_0)$$
$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)$$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\sin(M_0)$$
$$\left.-\frac{1}{2}\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)^2\right)\Bigg)e^6\Bigg)\delta^2+\Bigg(\cos(M_0)\,e$$
$$+\left(-\sin(M_0)^2+\cos(M_0)^2\right)e^2+\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)e^3+1+\Bigg(\sin(M_0)^4$$
$$-3\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$
$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Bigg)e^4+\Bigg(\cos(M_0)\Bigg(\frac{3}{2}\sin(M_0)^4$$
$$-\frac{9}{2}\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$
$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Bigg)$$
$$-\sin(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+$$
$$\sin(M_0)\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\right.$$
$$\left.-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)+\frac{7}{6}\sin(M_0)^3\cos(M_0)-\cos(M_0)^3\sin(M_0)\right)$$
$$+\cos(M_0)$$
$$\left.\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)\right)e^5$$
$$+\Bigg(\sin(M_0)\Bigg(\frac{2}{3}\cos(M_0)^2\sin(M_0)^3-\frac{1}{6}\sin(M_0)^5$$
$$-\left(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)\cos(M_0)-\Bigg($$
$$\sin(M_0)^4-3\cos(M_0)^2\sin(M_0)^2+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$
$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Bigg)\sin(M_0)$$

$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)^2-\Bigg($$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)$$
$$+\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\cos(M_0)\Bigg)\sin(M_0)\Bigg)+\cos(M_0)\Bigg(\frac{1}{6}\cos(M_0)\sin(M_0)^4$$
$$-\left(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)\sin(M_0)$$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\sin(M_0)\Bigg)+\cos(M_0)\Bigg(\frac{25}{24}\cos(M_0)\sin(M_0)^4$$
$$-\frac{3}{2}\cos(M_0)^3\sin(M_0)^2+\cos(M_0)\Bigg(\frac{3}{2}\sin(M_0)^4-\frac{9}{2}\cos(M_0)^2\sin(M_0)^2$$
$$+\cos(M_0)\left(-4\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)$$
$$-\sin(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\Bigg)$$
$$-\sin(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+$$
$$\sin(M_0)\Bigg(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)$$
$$-\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)+\frac{7}{6}\sin(M_0)^3\cos(M_0)-\cos(M_0)^3\sin(M_0)\Bigg)$$
$$+\cos(M_0)$$
$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)$$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\sin(M_0)$$
$$-\frac{1}{2}\left(-\frac{7}{2}\sin(M_0)^2\cos(M_0)+\cos(M_0)^3\right)\sin(M_0)^2\Bigg)-\sin(M_0)\Bigg(\frac{1}{120}\sin(M_0)^5$$
$$-\frac{1}{2}\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)^2-\frac{1}{2}\cos(M_0)^2\sin(M_0)^3$$
$$+\cos(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+$$
$$\sin(M_0)$$
$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)\Bigg)\Bigg)e^6$$
$$\Bigg)\delta+M_0+\left(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)e^4$$

$$+\sin(M_0)\,e+\Bigg($$
$$\cos(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+\sin(M_0)$$
$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)\Bigg)e^5$$
$$+\cos(M_0)\sin(M_0)\,e^2+\Bigg(\sin(M_0)\Bigg(\frac{1}{6}\cos(M_0)\sin(M_0)^4$$
$$-\left(-\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)\sin(M_0)$$
$$-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\cos(M_0)\sin(M_0)\Bigg)+\cos(M_0)\Bigg(\frac{1}{120}\sin(M_0)^5$$
$$-\frac{1}{2}\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)^2-\frac{1}{2}\cos(M_0)^2\sin(M_0)^3$$
$$+\cos(M_0)\left(-\frac{3}{2}\sin(M_0)^3\cos(M_0)+\cos(M_0)\left(-\frac{2}{3}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\right)+$$
$$\sin(M_0)$$
$$\left(-\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)\sin(M_0)+\frac{1}{24}\sin(M_0)^4-\frac{1}{2}\cos(M_0)^2\sin(M_0)^2\right)\Bigg)\Bigg)e^6$$
$$+\left(-\frac{1}{2}\sin(M_0)^3+\cos(M_0)^2\sin(M_0)\right)e^3$$

cost(%)

371 *additions* + 1300 *multiplications* + 742 *functions* + *assignments*

As we can see, expansions of useful order will generate large expressions. However, the appearance of these expressions can be considerably simplified by converting to Fourier form. For example,

collect(combine(%%, *trig*), [δ, *e*])

$$\phi=\Bigg(\left(-\frac{1}{24}\sin(2\,M_0)-\frac{243}{40}\sin(6\,M_0)+\frac{32}{15}\sin(4\,M_0)\right)e^6$$
$$+\left(-\frac{3125}{768}\sin(5\,M_0)+\frac{243}{256}\sin(3\,M_0)-\frac{1}{384}\sin(M_0)\right)e^5+\left(\frac{1}{3}\sin(2\,M_0)-\frac{8}{3}\sin(4\,M_0)\right)e^4$$
$$+\left(-\frac{27}{16}\sin(3\,M_0)+\frac{1}{16}\sin(M_0)\right)e^3-e^2\sin(2\,M_0)-\frac{1}{2}\sin(M_0)\,e\Bigg)\delta^2+\Bigg($$
$$\left(-\frac{16}{15}\cos(4\,M_0)+\frac{81}{40}\cos(6\,M_0)+\frac{1}{24}\cos(2\,M_0)\right)e^6$$
$$+\left(-\frac{81}{128}\cos(3\,M_0)+\frac{625}{384}\cos(5\,M_0)+\frac{1}{192}\cos(M_0)\right)e^5+\left(\frac{4}{3}\cos(4\,M_0)-\frac{1}{3}\cos(2\,M_0)\right)e^4$$

$$+\left(-\frac{1}{8}\cos(M_0)+\frac{9}{8}\cos(3\,M_0)\right)e^3+\cos(2\,M_0)\,e^2+\cos(M_0)\,e+1\Bigg)\delta$$
$$+\left(\frac{1}{48}\sin(2\,M_0)-\frac{4}{15}\sin(4\,M_0)+\frac{27}{80}\sin(6\,M_0)\right)e^6$$
$$+\left(\frac{125}{384}\sin(5\,M_0)-\frac{27}{128}\sin(3\,M_0)+\frac{1}{192}\sin(M_0)\right)e^5+\left(\frac{1}{3}\sin(4\,M_0)-\frac{1}{6}\sin(2\,M_0)\right)e^4$$
$$+\left(\frac{3}{8}\sin(3\,M_0)-\frac{1}{8}\sin(M_0)\right)e^3+\frac{1}{2}e^2\sin(2\,M_0)+\sin(M_0)\,e+M_0$$

```
cost(%)
```

37 *additions* + 125 *multiplications* + 36 *functions* + *assignments*

Characterizing the error term for a given time span

Introduction

Suppose we have a specified error bound η. We would like to say something about the approximation orders N_e and N_t required to get the error ε less than η as a function of the time interval μ. That is, find N such that $\varepsilon(N_e, N_t) < \eta$.

First, we write a function which returns an approximation of the solution of the Kepler equation, incorporating expansions in both eccentricity and time.

```
KeplerApprox := proc(ϕ::name, e::name, M::name, μ::name, Ne::posint, NM::posint)
local C, tmp, cpu;
    cpu := time( );
    trialsoln(ϕ, e, C, Ne);
    expansion(subs(%, Kepler(ϕ, e, M)), e, Ne);
    xsolve(%, e, C);
    subs(%, %%%);
    tmp := expansion(subs(M = M[0] + μ, %), μ, NM);
    if assigned(verbosity) and 0 < verbosity then
        print(`cost(raw)` = cost(tmp));
        tmp := collect(combine(tmp, trig), [μ, e]);
        print(`cost(Fourier)` = cost(tmp));
        print(`series expansion cpu` = time( ) − cpu);
        tmp
    else collect(combine(tmp, trig), [μ, e])
    fi
end
```

For example, KeplerApprox(ϕ, e, M, μ, 5, 3) yields

$$\phi=\left(\left(\frac{243}{256}\cos(3\,M_0)-\frac{15625}{2304}\cos(5\,M_0)-\frac{1}{1152}\cos(M_0)\right)e^5\right.$$

$$+\left(\frac{2}{9}\cos(2\,M_0)-\frac{32}{9}\cos(4\,M_0)\right)e^4+\left(-\frac{27}{16}\cos(3\,M_0)+\frac{1}{48}\cos(M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2$$
$$-\frac{1}{6}\cos(M_0)\,e\Bigg)\mu^3+\Bigg(\left(\frac{243}{256}\sin(3\,M_0)-\frac{3125}{768}\sin(5\,M_0)-\frac{1}{384}\sin(M_0)\right)e^5$$
$$+\left(-\frac{8}{3}\sin(4\,M_0)+\frac{1}{3}\sin(2\,M_0)\right)e^4+\left(-\frac{27}{16}\sin(3\,M_0)+\frac{1}{16}\sin(M_0)\right)e^3-e^2\sin(2\,M_0)$$
$$-\frac{1}{2}\sin(M_0)\,e\Bigg)\mu^2+\Bigg(\left(-\frac{81}{128}\cos(3\,M_0)+\frac{625}{384}\cos(5\,M_0)+\frac{1}{192}\cos(M_0)\right)e^5$$
$$+\left(-\frac{1}{3}\cos(2\,M_0)+\frac{4}{3}\cos(4\,M_0)\right)e^4+\left(-\frac{1}{8}\cos(M_0)+\frac{9}{8}\cos(3\,M_0)\right)e^3+\cos(2\,M_0)\,e^2$$
$$+\cos(M_0)\,e+1\Bigg)\mu+\left(\frac{125}{384}\sin(5\,M_0)-\frac{27}{128}\sin(3\,M_0)+\frac{1}{192}\sin(M_0)\right)e^5$$
$$+\left(\frac{1}{3}\sin(4\,M_0)-\frac{1}{6}\sin(2\,M_0)\right)e^4+\left(\frac{3}{8}\sin(3\,M_0)-\frac{1}{8}\sin(M_0)\right)e^3+\frac{1}{2}e^2\sin(2\,M_0)$$
$$+\sin(M_0)\,e+M_0$$

Here is a procedure to extract the highest-order term.

```
error_term := proc(e::name, M::name, μ::name, Ne::posint, NM::posint)
local ϕ;
    KeplerApprox(ϕ, e, M, μ, Ne, NM); lcoeff(rhs(%), μ)*μ^degree(rhs(%), μ)
end
```

For example, error_term(e, M, μ, 5, 3) yields

$$\Bigg(\left(-\frac{15625}{2304}\cos(5\,M_0)-\frac{1}{1152}\cos(M_0)+\frac{243}{256}\cos(3\,M_0)\right)e^5$$
$$+\left(\frac{2}{9}\cos(2\,M_0)-\frac{32}{9}\cos(4\,M_0)\right)e^4+\left(-\frac{27}{16}\cos(3\,M_0)+\frac{1}{48}\cos(M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2$$
$$-\frac{1}{6}\cos(M_0)\,e\Bigg)\mu^3$$

plot

Plot the coefficient of μ:

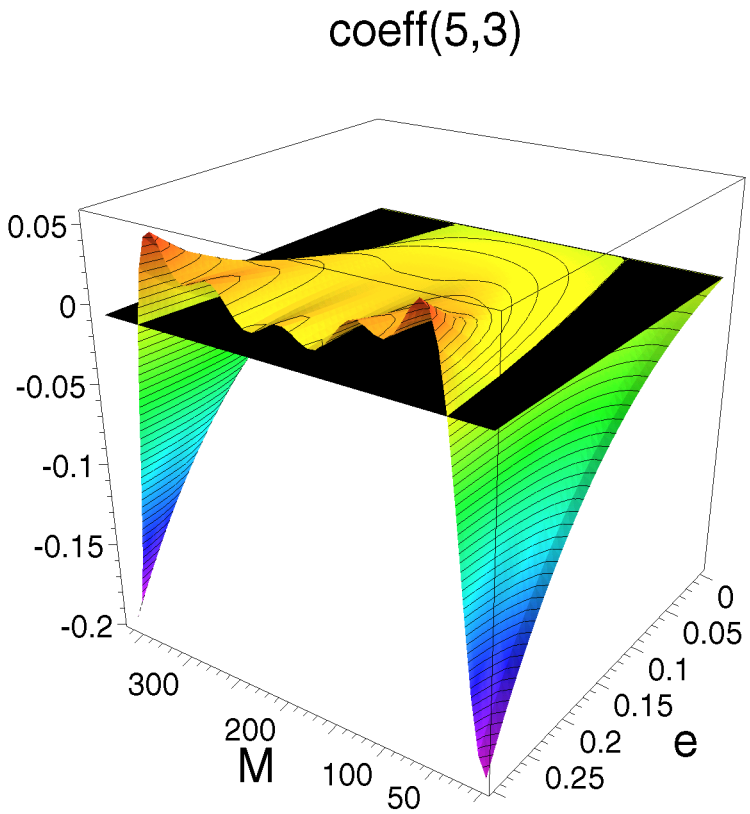
G := fn(subs($M_0 = M$, lcoeff(%, μ)), e, M)

$$G := (e, M) \rightarrow \left(\frac{243}{256}\cos(3\,M)-\frac{15625}{2304}\cos(5\,M)-\frac{1}{1152}\cos(M)\right)e^5$$
$$+\left(\frac{2}{9}\cos(2\,M)-\frac{32}{9}\cos(4\,M)\right)e^4+\left(-\frac{27}{16}\cos(3\,M)+\frac{1}{48}\cos(M)\right)e^3-\frac{2}{3}\cos(2\,M)\,e^2$$
$$-\frac{1}{6}\cos(M)\,e$$

$p0 := \text{plot3d}(0, M = 0 .. 360, e = 0 .. .3, style = patchnogrid, color = black)$

$p1 := \text{plot3d}(G(e, M\ deg), M = 0 .. 360, e = 0 .. .3, style = patchcontour, contours = 35, grid = [50, 50])$

$plots_{display3d}([p0, p1], labels = [\text{"M"}, \text{"e"}, \text{""}], orientation = [125, 65], title = \text{"coeff(5,3)"})$



The coefficients of μ^k

Introduction

The double series expansion (in eccentricity and in mean anomaly or time) is of the form $\phi = \sum_{k=0}^{N_t} \left(\sum_{n=0}^{N_e} C_{k,n}(M_0)\, e^n \right) \mu^k$. Define $F_{k,N_e}(e, M_0) = \sum_{n=0}^{N_e} C_{k,n}(M_0)\, e^n$ so that $\phi = \sum_{k=0}^{N_t} F_{k,N_e}(e, M_0)\, \mu^n$. The functions $F_{k,N_e}(e, M_0)$ will come up later, so we will characterize their typical behavior now.

```
F := proc(e, M, Ne::posint, n::posint) local μ; error_term(e, M, μ, Ne, n); coeff(%, μ, n) end
```

Variation of N_t

```
F(e, M, 3, 1)
```

$$\left(-\frac{1}{8}\cos(M_0)+\frac{9}{8}\cos(3\,M_0)\right)e^3+\cos(2\,M_0)\,e^2+\cos(M_0)\,e+1$$

F(*e*, *M*, 3, 2)

$$\left(-\frac{27}{16}\sin(3\,M_0)+\frac{1}{16}\sin(M_0)\right)e^3-e^2\sin(2\,M_0)-\frac{1}{2}\sin(M_0)\,e$$

F(*e*, *M*, 3, 3)

$$\left(-\frac{27}{16}\cos(3\,M_0)+\frac{1}{48}\cos(M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2-\frac{1}{6}\cos(M_0)\,e$$

F(*e*, *M*, 3, 4)

$$\left(\frac{81}{64}\sin(3\,M_0)-\frac{1}{192}\sin(M_0)\right)e^3+\frac{1}{3}e^2\sin(2\,M_0)+\frac{1}{24}\sin(M_0)\,e$$

F(*e*, *M*, 3, 5)

$$\left(-\frac{1}{960}\cos(M_0)+\frac{243}{320}\cos(3\,M_0)\right)e^3+\frac{2}{15}\cos(2\,M_0)\,e^2+\frac{1}{120}\cos(M_0)\,e$$

F(*e*, *M*, 3, 6)

$$\left(-\frac{243}{640}\sin(3\,M_0)+\frac{1}{5760}\sin(M_0)\right)e^3-\frac{2}{45}e^2\sin(2\,M_0)-\frac{1}{720}\sin(M_0)\,e$$

F(*e*, *M*, 3, 10)

$$\left(-\frac{2187}{358400}\sin(3\,M_0)+\frac{1}{29030400}\sin(M_0)\right)e^3-\frac{2}{14175}e^2\sin(2\,M_0)-\frac{1}{3628800}\sin(M_0)\,e$$

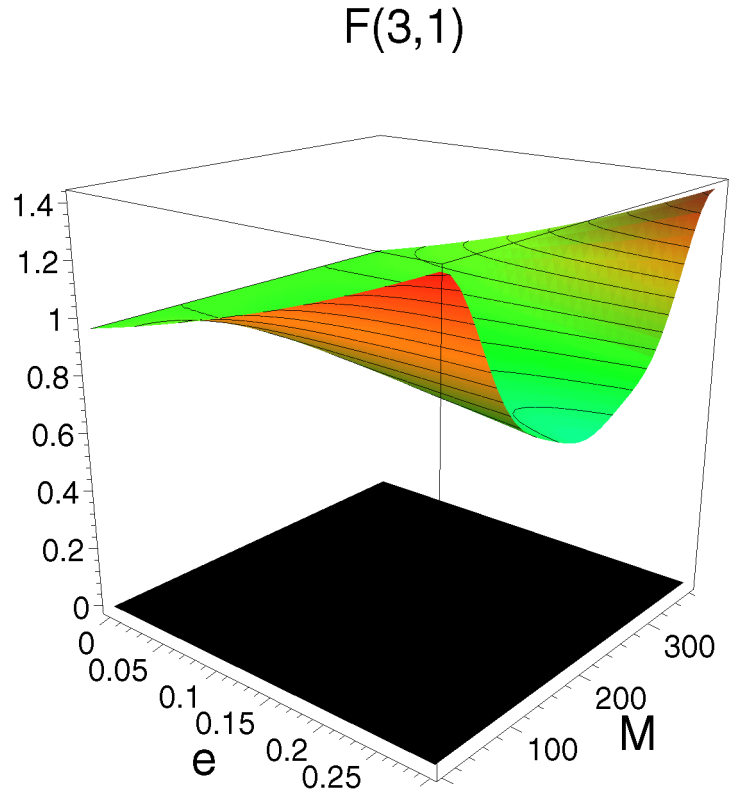
plots

```
plot_F := proc(emax, Ne::posint, n::posint)
local Fn, M, e, p0, p;
    p0 := plot3d(0, e = 0 .. emax, M = 0 .. 359.5, color = black);
    Fn := fn(subs(M[0] = M, F(e, M, Ne, n)), e, M);
    p := plot3d(Fn(e, M[0]*deg), e = 0 .. emax, M[0] = 0 .. 359.5, style = patchcontour,
        contours = 20, grid = [25, 25], args[4 .. nargs]);
    plots[display3d]([p, p0], labels = ["e", "M", ""], orientation = [-50, 70],
        lightmodel = light3, title = cat("F(", Ne, ",", n, ")"), args[4 .. nargs])
end
```

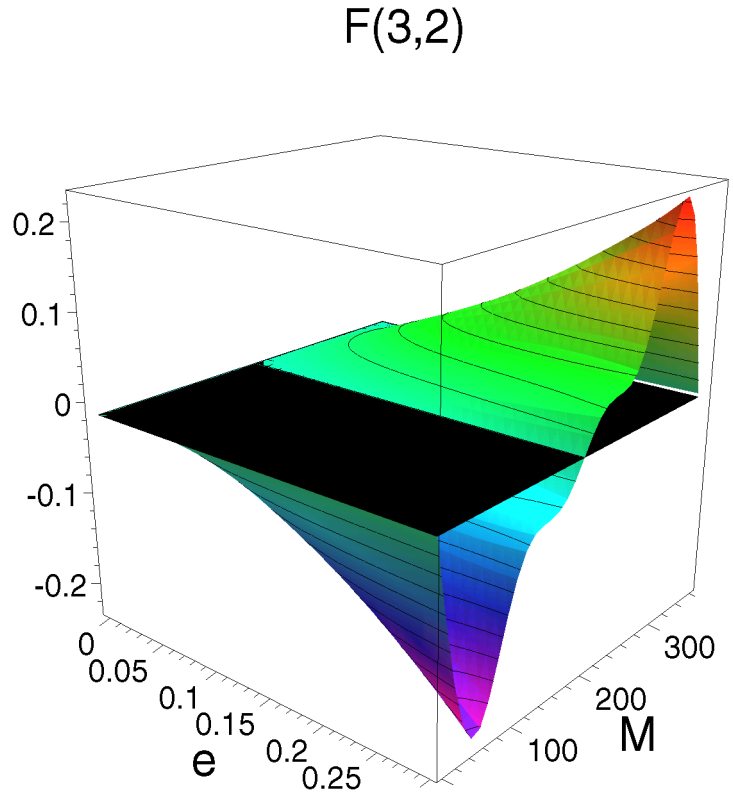
```
with(plotting)
```

```
emax := .3
```

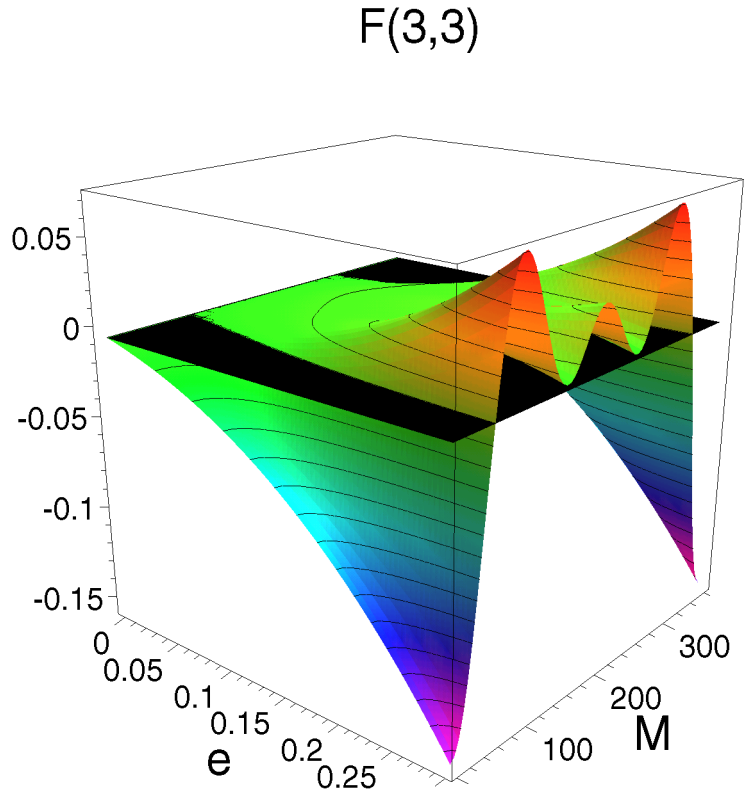
```
plot_F(emax, 3, 1, grid = [35, 35])
```



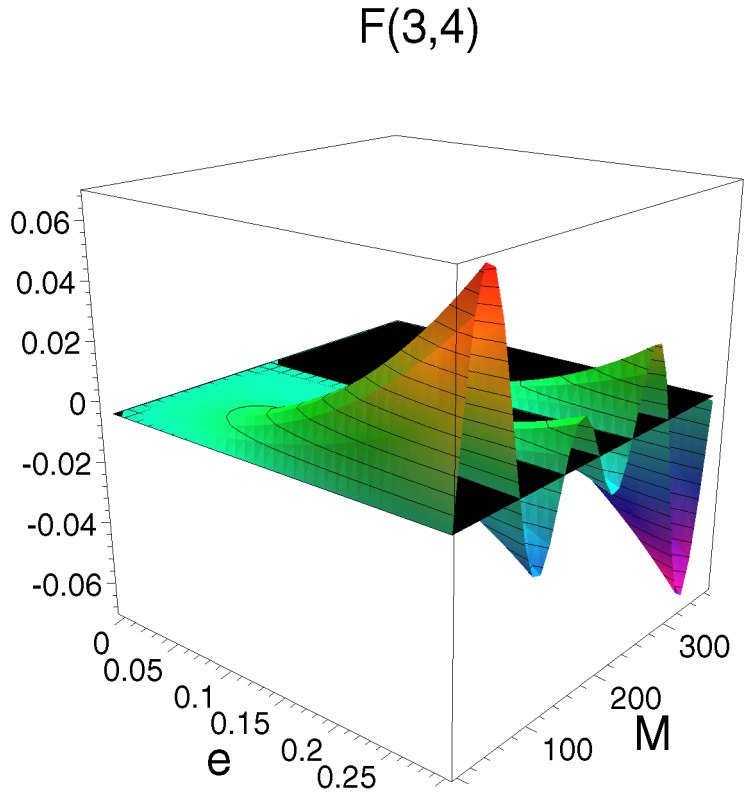
plot_F(*emax*, 3, 2, *grid* = [35, 35])



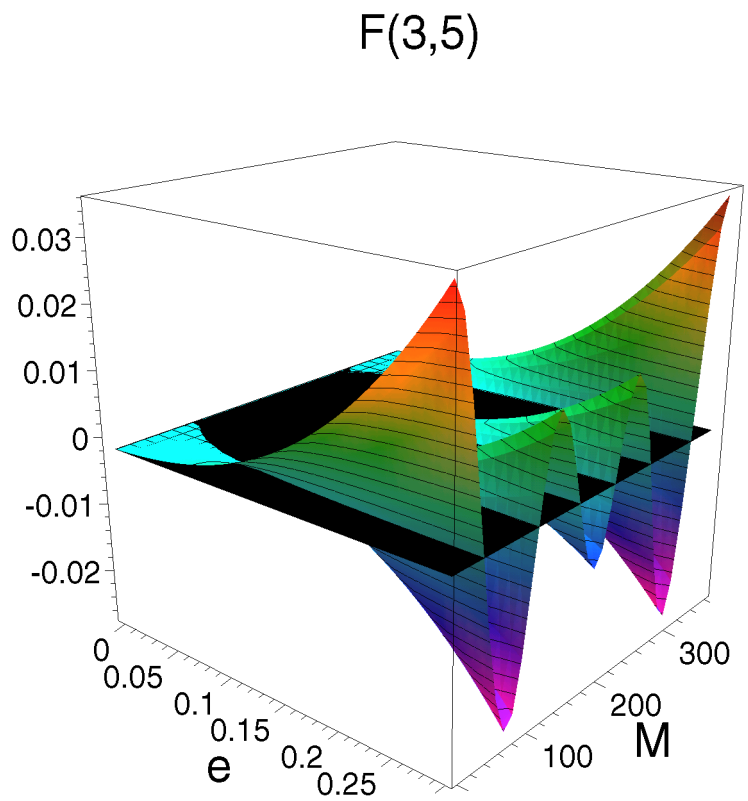
plot_F(*emax*, 3, 3, *grid* = [75, 75])



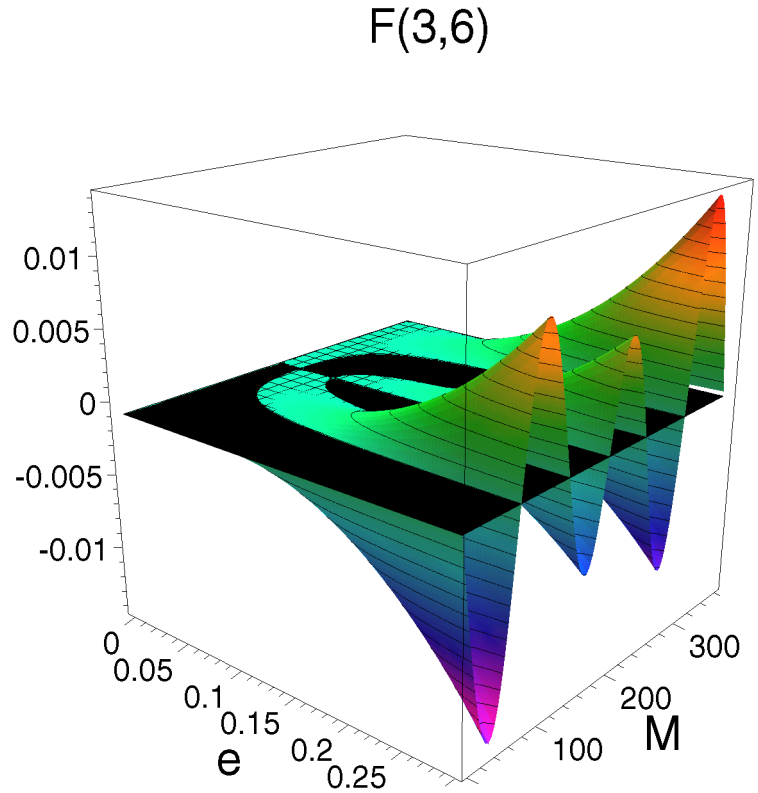

```
gifplot(%, "F33", "d:/dynamics/TimeSeriesOpt/")
jpgplot(%, "F33", "d:/dynamics/TimeSeriesOpt/")
psplot(%, "F33", "d:/dynamics/TimeSeriesOpt/")
plot_F(emax, 3, 4, grid = [35, 35])
```



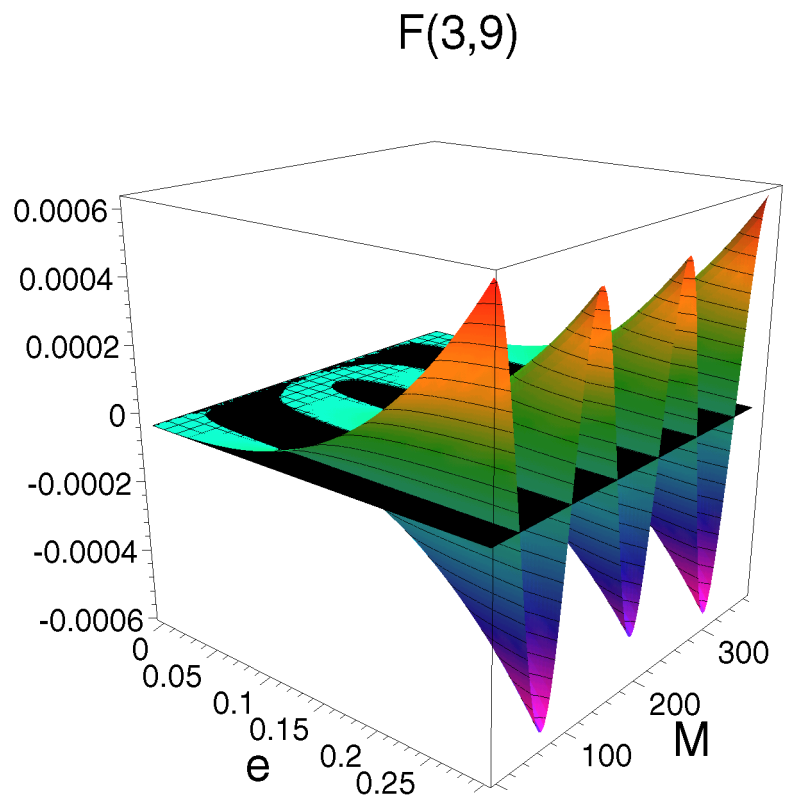
plot_F(*emax*, 3, 5, *grid* = [35, 35])



plot_F(*emax*, 3, 6, *grid* = [100, 100])




```
gifplot(%, "F36", "d:/dynamics/TimeSeriesOpt/")
jpgplot(%, "F36", "d:/dynamics/TimeSeriesOpt/")
psplot(%, "F36", "d:/dynamics/TimeSeriesOpt/")
plot_F(emax, 3, 9, grid = [100, 100])
```




```
gifplot(%, "F39", "d:/dynamics/TimeSeriesOpt/")
jpgplot(%, "F39", "d:/dynamics/TimeSeriesOpt/")
psplot(%, "F39", "d:/dynamics/TimeSeriesOpt/")
```

Variation of N_e

F(e, M, 1, 3)

$$-\frac{1}{6}\cos(M_0)\,e$$

F(e, M, 2, 3)

$$-\frac{2}{3}\cos(2\,M_0)\,e^2-\frac{1}{6}\cos(M_0)\,e$$

F(e, M, 3, 3)

$$\left(-\frac{27}{16}\cos(3\,M_0)+\frac{1}{48}\cos(M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2-\frac{1}{6}\cos(M_0)\,e$$

F(e, M, 4, 3)

$$\left(\frac{2}{9}\cos(2\,M_0)-\frac{32}{9}\cos(4\,M_0)\right)e^4+\left(-\frac{27}{16}\cos(3\,M_0)+\frac{1}{48}\cos(M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2-\frac{1}{6}\cos(M_0)\,e$$

F(e, M, 5, 3)

$$\left(\frac{243}{256}\cos(3\,M_0) - \frac{15625}{2304}\cos(5\,M_0) - \frac{1}{1152}\cos(M_0)\right)e^5 + \left(\frac{2}{9}\cos(2\,M_0) - \frac{32}{9}\cos(4\,M_0)\right)e^4 + \left(-\frac{27}{16}\cos(3\,M_0) + \frac{1}{48}\cos(M_0)\right)e^3 - \frac{2}{3}\cos(2\,M_0)\,e^2 - \frac{1}{6}\cos(M_0)\,e$$

verbosity := 1

F(e, M, 6, 3)

899 *additions* + 2957 *multiplications* + 1756 *functions* + *assignments*

49 *additions* + 170 *multiplications* + 48 *functions* + *assignments*

8.254

$$\left(\frac{128}{45}\cos(4\,M_0) - \frac{1}{36}\cos(2\,M_0) - \frac{243}{20}\cos(6\,M_0)\right)e^6 + \left(-\frac{15625}{2304}\cos(5\,M_0) + \frac{243}{256}\cos(3\,M_0) - \frac{1}{1152}\cos(M_0)\right)e^5 + \left(\frac{2}{9}\cos(2\,M_0) - \frac{32}{9}\cos(4\,M_0)\right)e^4 + \left(\frac{1}{48}\cos(M_0) - \frac{27}{16}\cos(3\,M_0)\right)e^3 - \frac{2}{3}\cos(2\,M_0)\,e^2 - \frac{1}{6}\cos(M_0)\,e$$

F(e, M, 7, 3)

3005 *additions* + 10202 *multiplications* + 6045 *functions* + *assignments*

65 *additions* + 226 *multiplications* + 64 *functions* + *assignments*

30.070

$$\left(\frac{1}{55296}\cos(M_0) - \frac{5764801}{276480}\cos(7\,M_0) + \frac{390625}{55296}\cos(5\,M_0) - \frac{2187}{10240}\cos(3\,M_0)\right)e^7 + \left(\frac{128}{45}\cos(4\,M_0) - \frac{1}{36}\cos(2\,M_0) - \frac{243}{20}\cos(6\,M_0)\right)e^6 + \left(-\frac{15625}{2304}\cos(5\,M_0) + \frac{243}{256}\cos(3\,M_0) - \frac{1}{1152}\cos(M_0)\right)e^5 + \left(\frac{2}{9}\cos(2\,M_0) - \frac{32}{9}\cos(4\,M_0)\right)e^4 + \left(\frac{1}{48}\cos(M_0) - \frac{27}{16}\cos(3\,M_0)\right)e^3 - \frac{2}{3}\cos(2\,M_0)\,e^2 - \frac{1}{6}\cos(M_0)\,e$$

F(e, M, 8, 3)

9742 *additions* + 33991 *multiplications* + 19997 *functions* + *assignments*

81 *additions* + 290 *multiplications* + 80 *functions* + *assignments*

136.473

$$\left(-\frac{32768}{945}\cos(8\,M_0)-\frac{128}{135}\cos(4\,M_0)+\frac{1}{540}\cos(2\,M_0)+\frac{2187}{140}\cos(6\,M_0)\right)e^8$$
$$+\left(\frac{1}{55296}\cos(M_0)-\frac{5764801}{276480}\cos(7\,M_0)+\frac{390625}{55296}\cos(5\,M_0)-\frac{2187}{10240}\cos(3\,M_0)\right)e^7$$
$$+\left(\frac{128}{45}\cos(4\,M_0)-\frac{1}{36}\cos(2\,M_0)-\frac{243}{20}\cos(6\,M_0)\right)e^6$$
$$+\left(-\frac{15625}{2304}\cos(5\,M_0)+\frac{243}{256}\cos(3\,M_0)-\frac{1}{1152}\cos(M_0)\right)e^5$$
$$+\left(\frac{2}{9}\cos(2\,M_0)-\frac{32}{9}\cos(4\,M_0)\right)e^4+\left(\frac{1}{48}\cos(M_0)-\frac{27}{16}\cos(3\,M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2$$
$$-\frac{1}{6}\cos(M_0)\,e$$

F(e, M, 9, 3)

30720 *additions* + 109643 *multiplications* + 64000 *functions* + *assignments*

101 *additions* + 362 *multiplications* + 100 *functions* + *assignments*

341.442

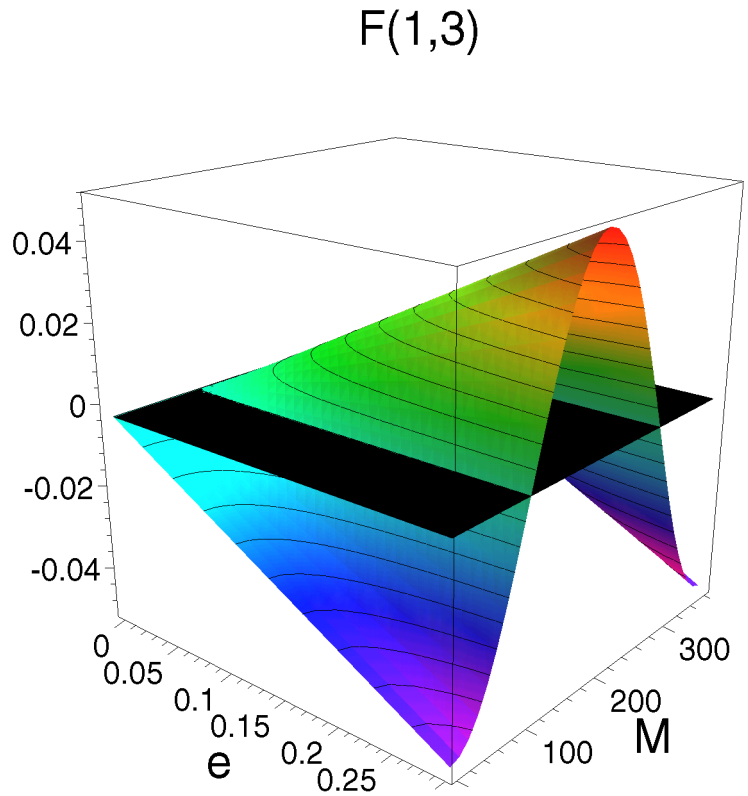
$$\left(-\frac{129140163}{2293760}\cos(9\,M_0)+\frac{2187}{81920}\cos(3\,M_0)-\frac{1}{4423680}\cos(M_0)+\frac{282475249}{8847360}\cos(7\,M_0)\right.$$
$$\left.-\frac{9765625}{3096576}\cos(5\,M_0)\right)e^9$$
$$+\left(-\frac{32768}{945}\cos(8\,M_0)-\frac{128}{135}\cos(4\,M_0)+\frac{1}{540}\cos(2\,M_0)+\frac{2187}{140}\cos(6\,M_0)\right)e^8$$
$$+\left(\frac{1}{55296}\cos(M_0)-\frac{5764801}{276480}\cos(7\,M_0)+\frac{390625}{55296}\cos(5\,M_0)-\frac{2187}{10240}\cos(3\,M_0)\right)e^7$$
$$+\left(\frac{128}{45}\cos(4\,M_0)-\frac{1}{36}\cos(2\,M_0)-\frac{243}{20}\cos(6\,M_0)\right)e^6$$
$$+\left(-\frac{15625}{2304}\cos(5\,M_0)+\frac{243}{256}\cos(3\,M_0)-\frac{1}{1152}\cos(M_0)\right)e^5$$
$$+\left(\frac{2}{9}\cos(2\,M_0)-\frac{32}{9}\cos(4\,M_0)\right)e^4+\left(\frac{1}{48}\cos(M_0)-\frac{27}{16}\cos(3\,M_0)\right)e^3-\frac{2}{3}\cos(2\,M_0)\,e^2$$
$$-\frac{1}{6}\cos(M_0)\,e$$

verbosity := 0

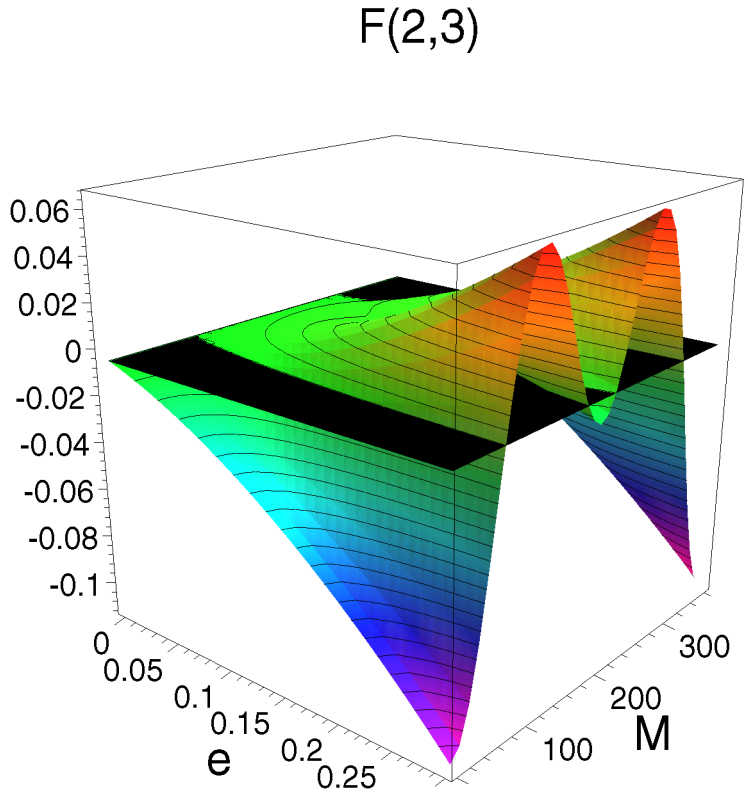
plots

emax := .3

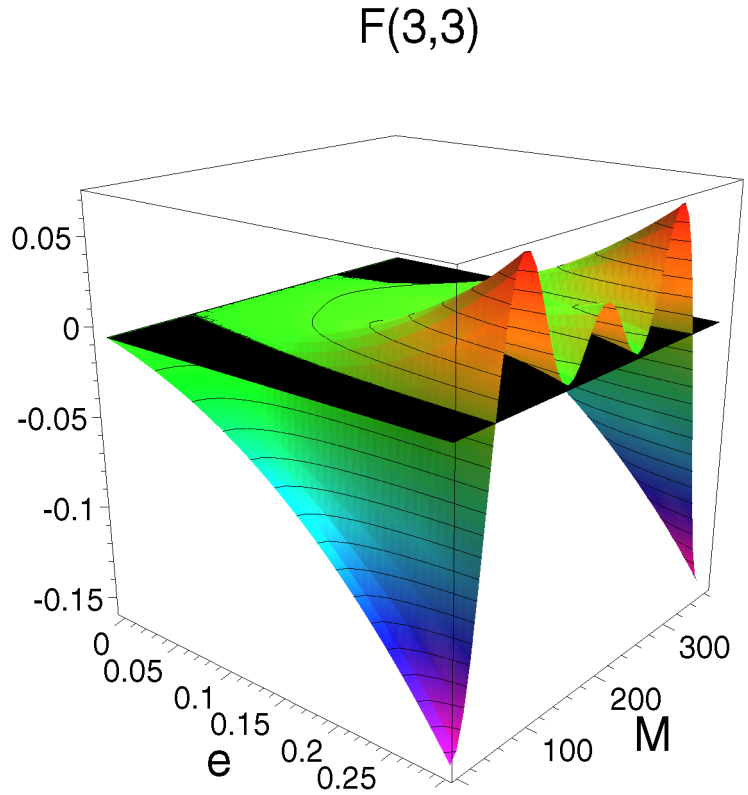
plot_F(*emax*, 1, 3, *grid* = [35, 35])



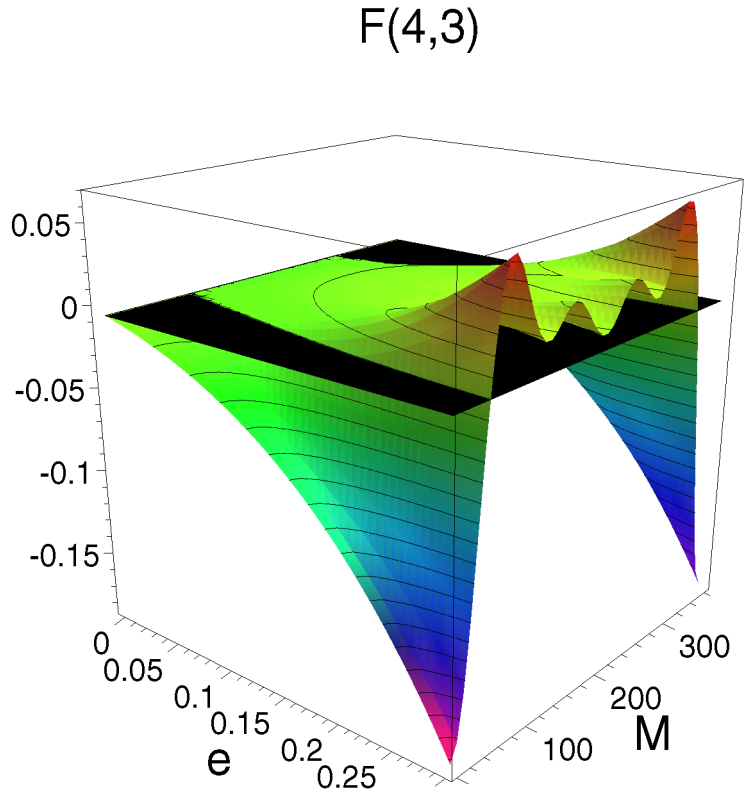
plot_F(*emax*, 2, 3, *grid* = [35, 35])



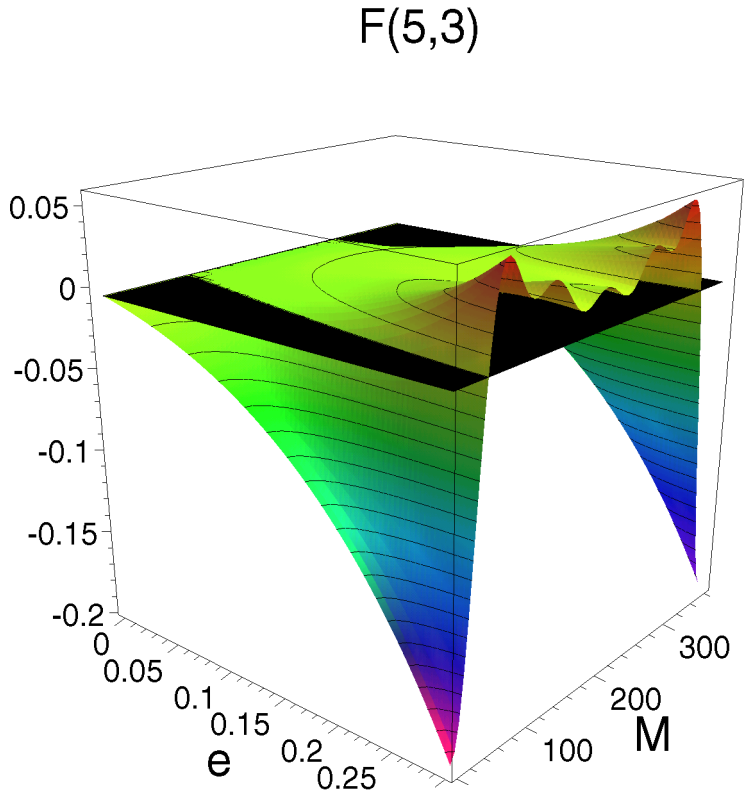
plot_F(*emax*, 3, 3, *grid* = [50, 50])



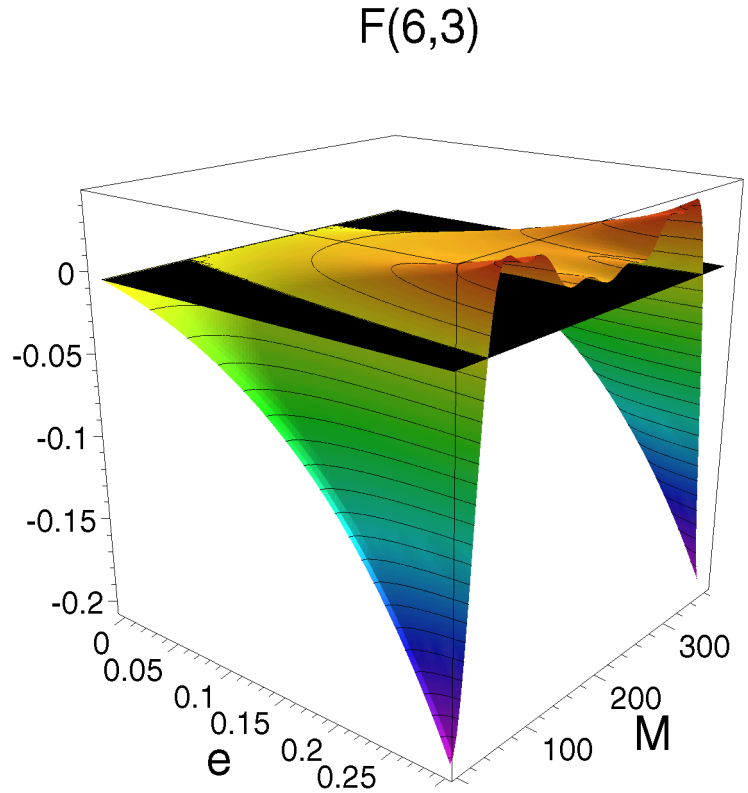
plot_F(*emax*, 4, 3, *grid* = [50, 50], *lightmodel* = *light2*)



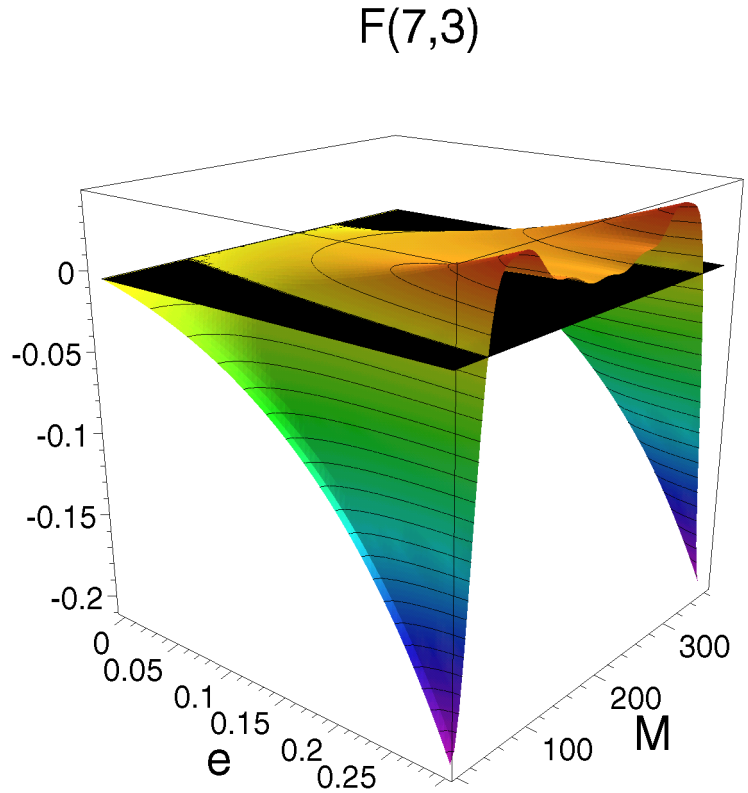
plot_F(*emax*, 5, 3, *grid* = [75, 75], *lightmodel* = *light2*)



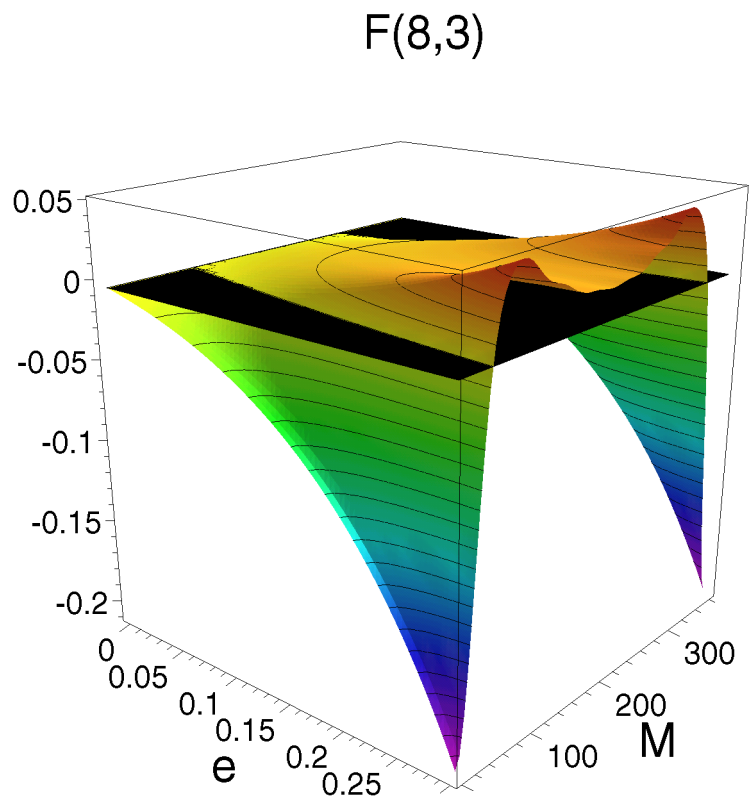
plot_F(*emax*, 6, 3, *grid* = [100, 100], *lightmodel* = *light4*)



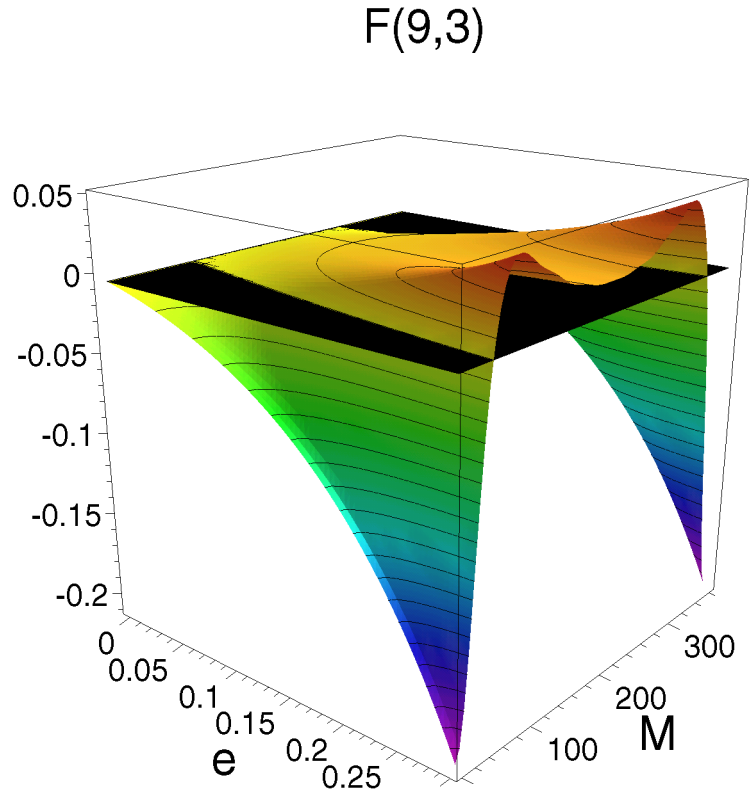

```
gifplot(%, "F63", "d:/dynamics/TimeSeriesOpt/")
jpgplot(%, "F63", "d:/dynamics/TimeSeriesOpt/")
psplot(%, "F63", "d:/dynamics/TimeSeriesOpt/")
plot_F(emax, 7, 3, grid = [75, 75], lightmodel = light4)
```



plot_F(*emax*, 8, 3, *grid* = [75, 75], *lightmodel* = *light4*)



plot_F(*emax*, 9, 3, *grid* = [75, 75], *lightmodel* = *light4*)




```
gifplot(%, "F93", "d:/dynamics/TimeSeriesOpt/")
jpgplot(%, "F93", "d:/dynamics/TimeSeriesOpt/")
psplot(%, "F93", "d:/dynamics/TimeSeriesOpt/")
```

Solving for the timespan

Now write a procedure that solves for the timespan μ when the error term is set equal to some maximum error ε, with μ in units of 2 π. We optimize the output in order to reduce numerical computation time.

```
timespan := proc(M, e, ε, Ne::posint, NM::posint)
    error_term(e, M, μ, Ne, NM);
    collect(expand(%, trig), [μ, e], factor);
    convert(%, horner, [μ, e]);
    ε / subs(M[0] = M, lcoeff(%, μ));
    abs(%)^(1 / NM) / (2*π);
    makeproc([optimize(%, tryhard)], parameters = [M, e, ε])
end
```

The output of this procedure is an optimized Maple procedure suitable for use in plotting. For example,

```
G := timespan(x, e, ε, 3, 2)
```

```
G := proc(x, e, ε)
local t1, t2;
```

```
    t1 := cos(x);
    t2 := 1 / 2*sqrt(abs(ε / ((−1 / 2 + (7 / 4*e + (−2 − 27 / 4*e*t1)*t1)*e)*sin(x)*e))) / π
end
```

$\mu(3, 2) = G(M, e, \varepsilon)$

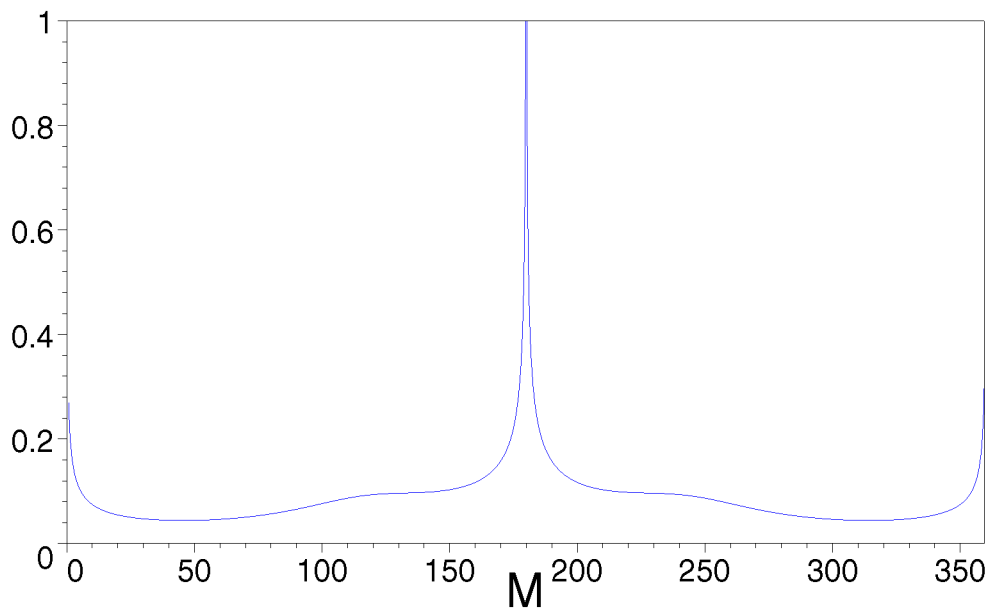
$$\mu(3, 2) = \frac{1}{2}\frac{\sqrt{\left|\frac{\varepsilon}{\left(-\frac{1}{2}+\left(\frac{7}{4}e+\left(-2-\frac{27}{4}e\cos(M)\right)\cos(M)\right)e\right)\sin(M)\,e}\right|}}{\pi}$$

Illustrative timespan plots

A 2D Slice

Here is a representative slice of $\mu(3, 2)$:

```
plot(G(M deg, .3, 1 deg), M = 0 .. 359.5, view = [0 .. 360, 0 .. 1], numpoints = 501)
```



Calculations

```
plot_timespan := proc(eps, emax, Ne::posint, Nt::posint)
local F, M, e, ε, p0, p;
    p0 := plot3d(0, e = 0 .. emax, M = 0 .. 359.5, color = black);
    F := timespan(M, e, ε, Ne, Nt);
    if assigned(verbosity) and 0 < verbosity then print(μ(Ne, Nt) = F(M, e, ε)) fi;
    p := plot3d(F(M*deg, e, eps), e = 0 .. emax, M = 0 .. 359.5, style = patchcontour,
        contours = 20, grid = [25, 25], args[5 .. nargs]);
    plots[display3d]([p, p0], labels = ["e", "M", ""], orientation = [−50, 70],
        lightmodel = light3, titlefont = [SYMBOL, 18], title = cat("m(", Ne, ",", Nt, ")"),
        args[5 .. nargs])
end
```

$verbosity := 1$

$emax := .3$

$\varepsilon := 1\ deg$

$p31 :=$ plot_timespan(ε, $emax$, 3, 1)

cost(raw) = 10 *additions* + 26 *multiplications* + 12 *functions* + 13 *subscripts* + *assignments*

cost(Fourier) = 9 *additions* + 22 *multiplications* + 8 *functions* + 9 *subscripts* + *assignments*

series expansion cpu = .333

$$\mu(3, 1) = \frac{1}{2}\frac{\left|\frac{\varepsilon}{1+\left(\cos(M)+\left(2\cos(M)^2-1+\frac{1}{2}\cos(M)\,(-7+9\cos(M)^2)\,e\right)e\right)e}\right|}{\pi}$$

$p32 :=$ plot_timespan(ε, $emax$, 3, 2, $grid = [51, 51]$, $view = [0 .. emax, 0 .. 360, 0 .. 1]$)

cost(raw) = 18 *additions* + 50 *multiplications* + 24 *functions* + 25 *subscripts* + *assignments*

cost(Fourier) =

13 *additions* + 35 *multiplications* + 12 *functions* + 13 *subscripts* + *assignments*

series expansion cpu = .666

$$\mu(3, 2) = \frac{1}{2}\frac{\sqrt{\left|\frac{\varepsilon}{\left(-\frac{1}{2}+\left(-2\cos(M)+\left(-\frac{27}{4}\cos(M)^2+\frac{7}{4}\right)e\right)e\right)\sin(M)\,e}\right|}}{\pi}$$

$p52 :=$ plot_timespan(ε, $emax$, 5, 2, $grid = [51, 51]$, $view = [0 .. emax, 0 .. 360, 0 .. 1]$)

cost(raw) = 119 *additions* + 398 *multiplications* + 221 *functions* + *assignments*

cost(Fourier) = 28 *additions* + 89 *multiplications* + 27 *functions* + *assignments*

series expansion cpu = 1.964

$$\mu(5, 2) = \frac{1}{2}\,\mathrm{sqrt}\left(\left|\varepsilon\,/\left(\left(-\frac{1}{2}+\left(-2\cos(M)+\left(-\frac{27}{4}\cos(M)^2+\frac{7}{4}+\left(-\frac{2}{3}\cos(M)\,(-17+32\cos(M)^2)+\left(-\frac{241}{48}+\left(\frac{421}{8}-\frac{3125}{48}\cos(M)^2\right)\cos(M)^2\right)e\right)e\right)e\right)\sin(M)\,e\right)\right|\right)/\,\pi$$

$p33 :=$ plot_timespan(ε, $emax$, 3, 3, $grid = [100, 100]$, $view = [0 .. emax, 0 .. 360, 0 .. 1]$)

cost(raw) = 27 *additions* + 77 *multiplications* + 36 *functions* + *assignments*

cost(Fourier) = 17 *additions* + 50 *multiplications* + 16 *functions* + *assignments*

series expansion cpu = 1.016

$$\mu(3,3) = \frac{1}{2}\left|\frac{\varepsilon}{\left(-\frac{1}{6}\cos(M)+\left(-\frac{4}{3}\cos(M)^2+\frac{2}{3}-\frac{1}{12}\cos(M)\,(-61+81\cos(M)^2)\,e\right)e\right)e}\right|^{(1/3)} \Big/ \pi$$

p53 := plot_timespan(ε, *emax*, 5, 3, *grid* = [100, 100], *view* = [0 .. *emax*, 0 .. 360, 0 .. 1])

cost(raw) =

251 *additions* + 792 *multiplications* + 461 *functions* + 462 *subscripts* + *assignments*

cost(Fourier) =

37 *additions* + 122 *multiplications* + 36 *functions* + 37 *subscripts* + *assignments*

series expansion cpu = 4.553

$$\mu(5,3) = \frac{1}{2}\left|\varepsilon \Big/ \left(\left(-\frac{1}{6}\cos(M)+\left(-\frac{4}{3}\cos(M)^2+\frac{2}{3}+\left(\left(-\frac{27}{4}\cos(M)^2+\frac{61}{12}\right)\cos(M)+\left(-\frac{34}{9}-\frac{256}{9}\cos(M)^4+\frac{260}{9}\cos(M)^2+\left(-\frac{5293}{144}+\frac{10039}{72}\cos(M)^2-\frac{15625}{144}\cos(M)^4\right)\cos(M)\,e\right)e\right)e\right)e\right)\right|^{(1/3)} \Big/ \pi$$

p73 := plot_timespan(ε, *emax*, 7, 3, *grid* = [100, 100], *view* = [0 .. *emax*, 0 .. 360, 0 .. 1])

cost(raw) =

3005 *additions* + 10202 *multiplications* + 6045 *functions* + 6046 *subscripts* + *assignments*

cost(Fourier) =

65 *additions* + 226 *multiplications* + 64 *functions* + 65 *subscripts* + *assignments*

series expansion cpu = 65.437

$$\mu(7,3) = \frac{1}{2}\left|\varepsilon \Big/ \left(\left(-\frac{1}{6}\cos(M)+\left(\frac{2}{3}-\frac{4}{3}\cos(M)^2+\left(-\frac{1}{12}\cos(M)\,(-61+81\cos(M)^2)+\left(-\frac{34}{9}-\frac{256}{9}\cos(M)^4+\frac{260}{9}\cos(M)^2+\left(\left(-\frac{15625}{144}\cos(M)^4+\frac{10039}{72}\cos(M)^2-\frac{5293}{144}\right)\cos(M)+\left(-\frac{10868}{45}\cos(M)^2+\frac{676}{45}-\frac{1944}{5}\cos(M)^6+\frac{27268}{45}\cos(M)^4+\left(-\frac{1886081}{1440}\cos(M)^2+\frac{785881}{4320}-\frac{5764801}{4320}\cos(M)^6+\frac{391729}{160}\cos(M)^4\right)\cos(M)\,e\right)e\right)e\right)\right)\right)\right|^{(1/3)} \Big/ \pi$$

$$e\Big)e\Big)e\Big)e\Big)\Big|^{(1/3)}/\pi$$

p35 := plot_timespan(ε, *emax*, 3, 5, *grid* = [100, 100], *view* = [0 .. *emax*, 0 .. 360, 0 .. 1])

cost(raw) = 52 *additions* + 156 *multiplications* + 72 *functions* + 73 *subscripts* + *assignments*

cost(Fourier) =

25 *additions* + 83 *multiplications* + 24 *functions* + 25 *subscripts* + *assignments*

series expansion cpu = 3.574

$$\mu(3,5) = \frac{1}{2}\left|\frac{\varepsilon}{\left(\frac{1}{120}\cos(M) + \left(\frac{4}{15}\cos(M)^2 - \frac{2}{15} + \frac{1}{240}\cos(M)\,(729\cos(M)^2 - 547)\,e\right)e\right)e}\right|^{(1/5)}\Bigg/\pi$$

p55 := plot_timespan(ε, *emax*, 5, 5, *grid* = [100, 100], *view* = [0 .. *emax*, 0 .. 360, 0 .. 1])

cost(raw) = 826 *additions* + 2449 *multiplications* + 1470 *functions* + *assignments*

cost(Fourier) = 55 *additions* + 191 *multiplications* + 54 *functions* + *assignments*

series expansion cpu = 5.523

$$\mu(5,5) = \frac{1}{2}\Bigg|\varepsilon\Big/\Bigg(\Bigg(\frac{1}{120}\cos(M) + \Bigg(\frac{4}{15}\cos(M)^2 - \frac{2}{15} + \Bigg(\Bigg(\frac{243}{80}\cos(M)^2 - \frac{547}{240}\Bigg)\cos(M) + \Bigg(\frac{26}{9} + \frac{1024}{45}\cos(M)^4 - \frac{1028}{45}\cos(M)^2 + \Bigg(-\frac{246601}{1440}\cos(M)^2 + \frac{125761}{2880} + \frac{78125}{576}\cos(M)^4\Bigg)\cos(M)\,e\Bigg)e\Bigg)e\Bigg)e\Bigg)\Bigg|^{(1/5)}\Bigg/\pi$$

p75 := plot_timespan(ε, *emax*, 7, 5, *grid* = [100, 100], *view* = [0 .. *emax*, 0 .. 360, 0 .. 1])

cost(raw) = 14481 *additions* + 45416 *multiplications* + 27675 *functions* + 27676 *subscripts* + *assignments*

cost(Fourier) =

97 *additions* + 347 *multiplications* + 96 *functions* + 97 *subscripts* + *assignments*

series expansion cpu = 112.694

$$\mu(7,5) = \frac{1}{2}\Bigg|\varepsilon\Big/\Bigg(\Bigg(\frac{1}{120}\cos(M) + \Bigg(-\frac{2}{15} + \frac{4}{15}\cos(M)^2 + \Bigg(\frac{1}{240}\cos(M)\,(-547 + 729\cos(M)^2) + \Bigg(\frac{1024}{45}\cos(M)^4 - \frac{1028}{45}\cos(M)^2 + \frac{26}{9} + \Bigg(\Bigg(\frac{78125}{576}\cos(M)^4 + \frac{125761}{2880} - \frac{246601}{1440}\cos(M)^2\Bigg)\cos(M) + \Bigg(-\frac{5434}{225} + \frac{17496}{25}\cos(M)^6$$

$$-\frac{240292}{225}\cos(M)^4+\frac{92672}{225}\cos(M)^2$$
$$+\left(\frac{9720661}{3200}\cos(M)^2-\frac{34735339}{86400}+\frac{282475249}{86400}\cos(M)^6-\frac{168846239}{28800}\cos(M)^4\right)\cos(M)\,e$$
$$\Big)e\Big)e\Big)e\Big)e\Big)e\Big)e\Big)\Big|^{(1/5)}/\pi$$

p77 := plot_timespan(ε, *emax*, 7, 7, *grid* = [150, 150], *view* = [0 .. *emax*, 0 .. 360, 0 .. 1])

cost(raw) = 49294 *additions* + 148392 *multiplications* + 90878 *functions* + *assignments*

cost(Fourier) = 129 *additions* + 472 *multiplications* + 128 *functions* + *assignments*

series expansion cpu = 303.908

$$\mu(7,7)=\frac{1}{2}\Big|\varepsilon/\Big(\Big(-\frac{1}{5040}\cos(M)+\Big(\frac{4}{315}-\frac{8}{315}\cos(M)^2+\Big(\Big(-\frac{8192}{945}\cos(M)^4$$
$$+\frac{1640}{189}\cos(M)^2-\frac{1028}{945}$$
$$+\left(-\frac{104976}{175}\cos(M)^6+\frac{4284296}{4725}\cos(M)^4-\frac{1627096}{4725}\cos(M)^2+\frac{92672}{4725}\right)e^2\Big)e+\Big(\frac{703}{1440}$$
$$-\frac{729}{1120}\cos(M)^2+\Big(\frac{6125659}{60480}\cos(M)^2-\frac{1953125}{24192}\cos(M)^4-\frac{3084973}{120960}+$$
$$\left(\frac{8175809461}{1209600}\cos(M)^4-\frac{1977326743}{518400}\cos(M)^6+\frac{1609482421}{3628800}-\frac{4164298321}{1209600}\cos(M)^2\right)e^2\Big)$$
$$e^2\Big)\cos(M)\Big)e\Big)e\Big)e\Big)\Big|^{(1/7)}/\pi$$

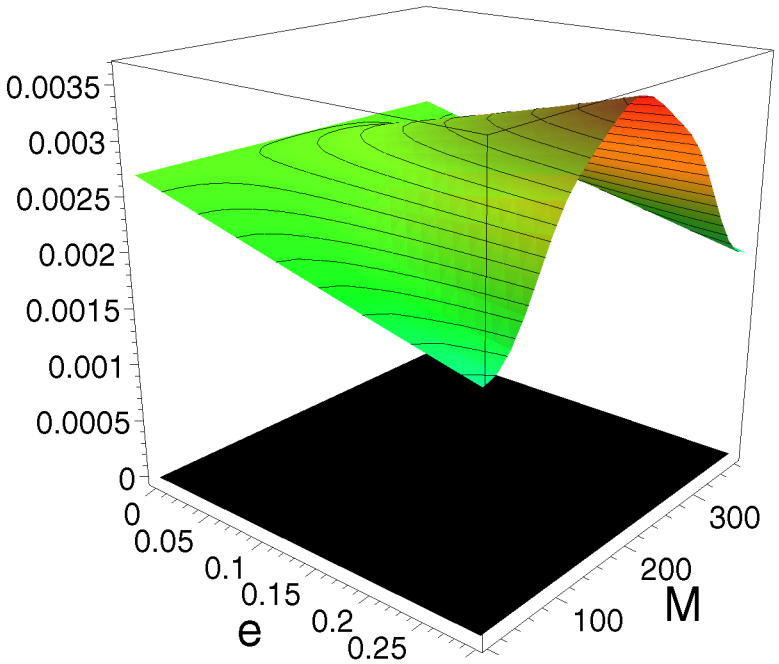
save *p31*, *p32*, *p52*, *p33*, *p53*, *p73*, *p35*, *p55*, *p75*, *p77*, "d:/dynamics/TimeSeriesOpt/timespan_3D_plots.m"

?

Images

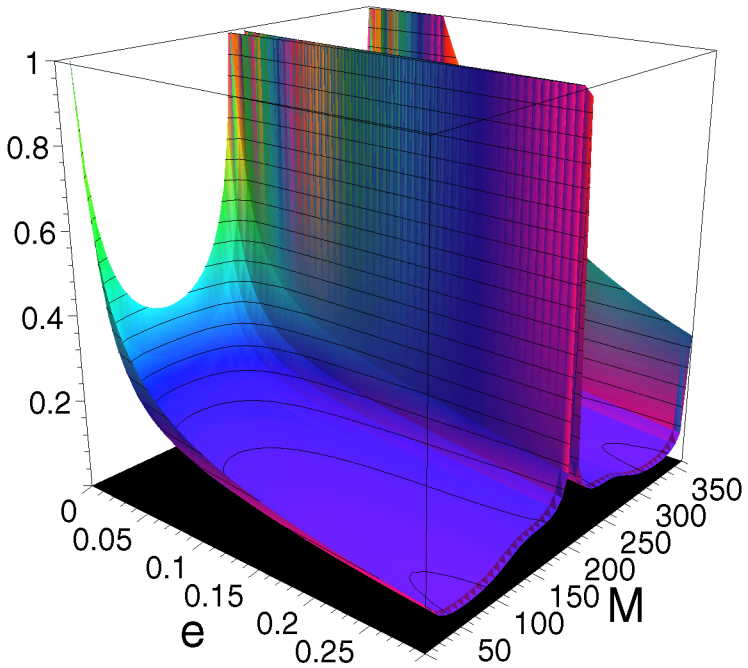
p31

μ(3,1)



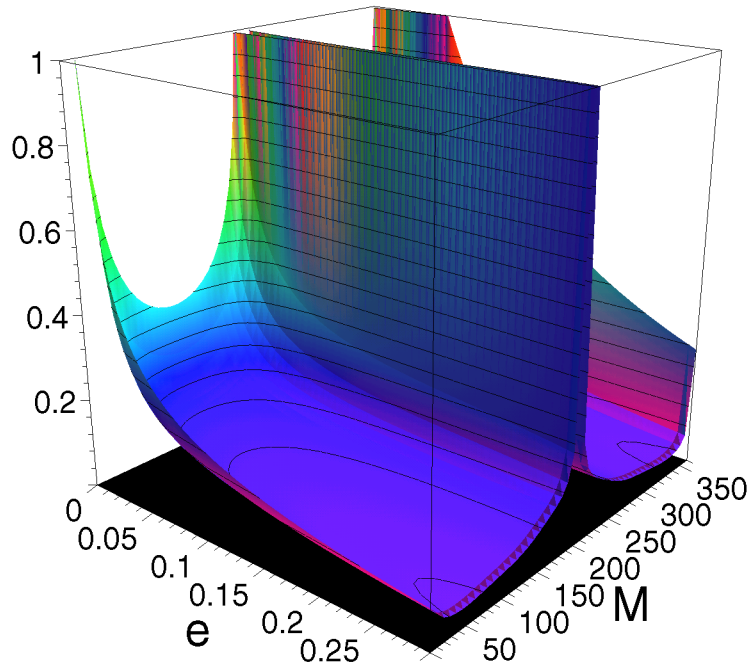
p32

μ(3,2)



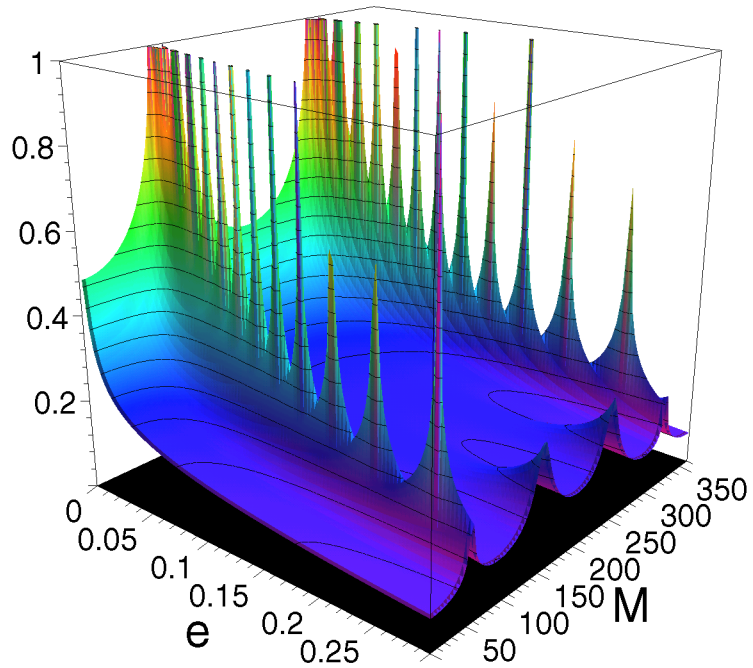
p52

μ(5,2)



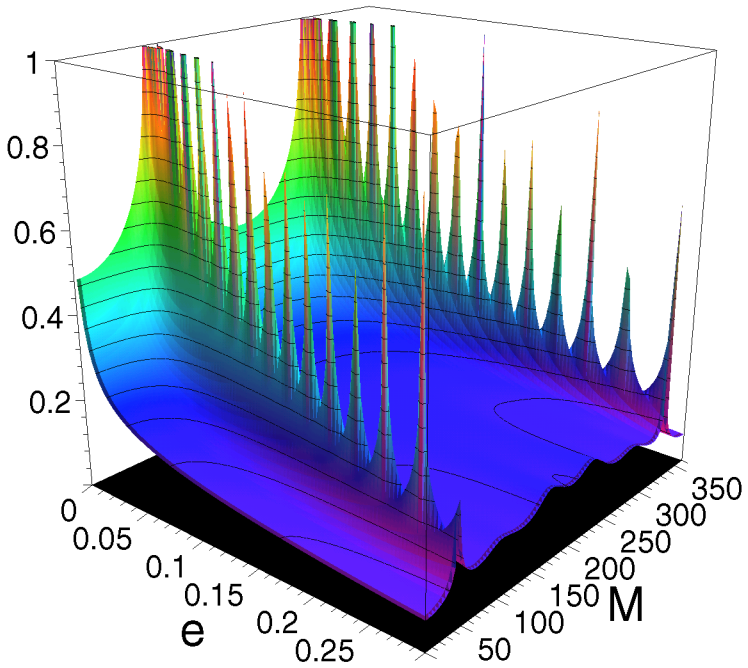
p33

μ(3,3)



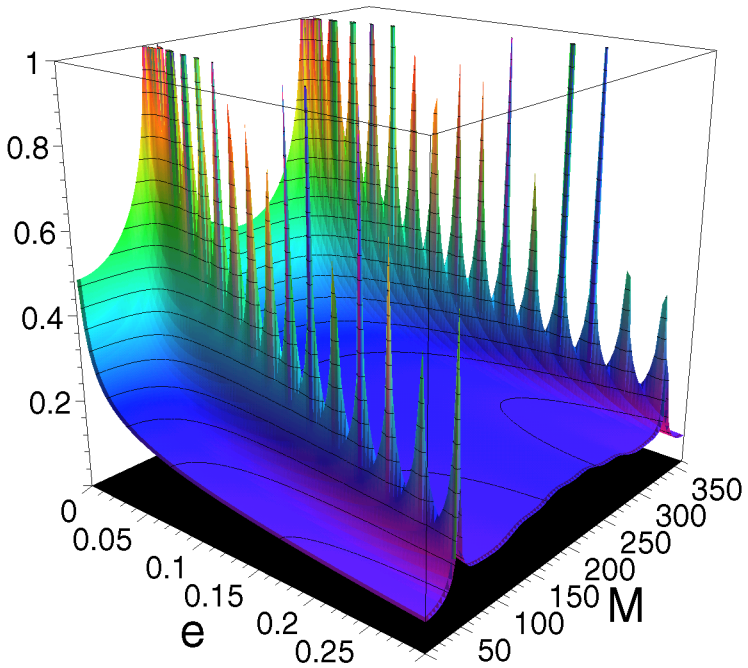
p53

μ(5,3)



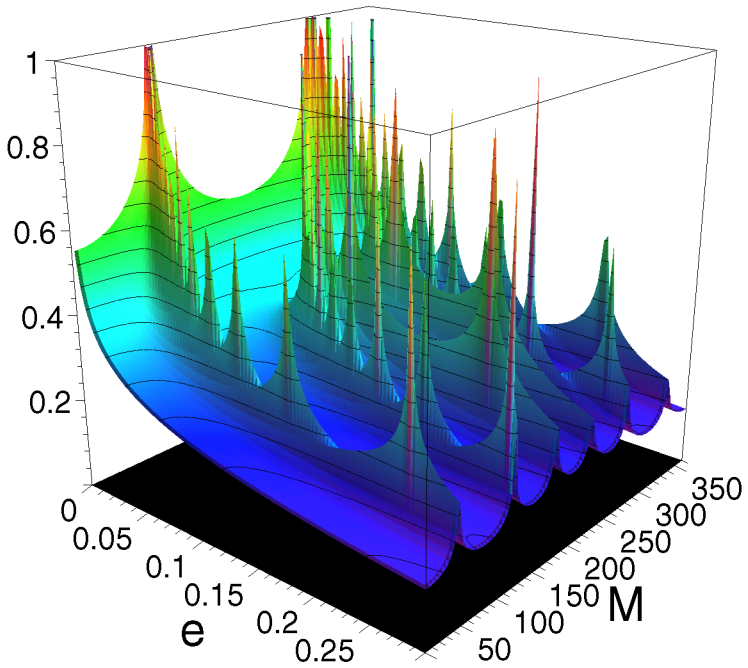
p73

μ(7,3)



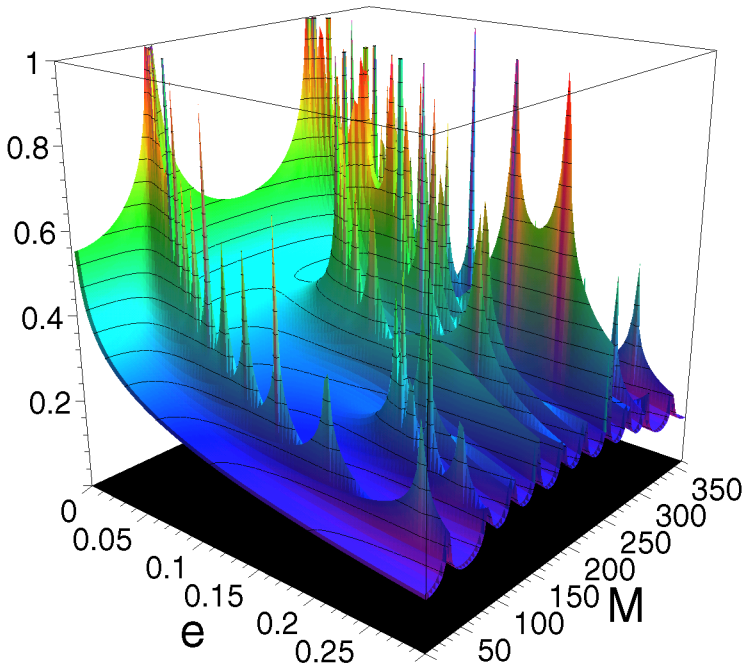
p35

μ(3,5)



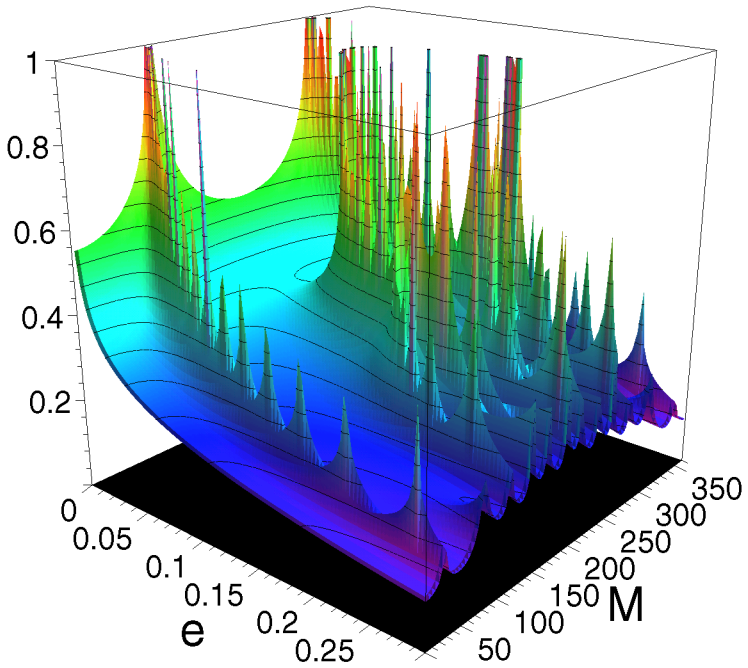
p55

μ(5,5)



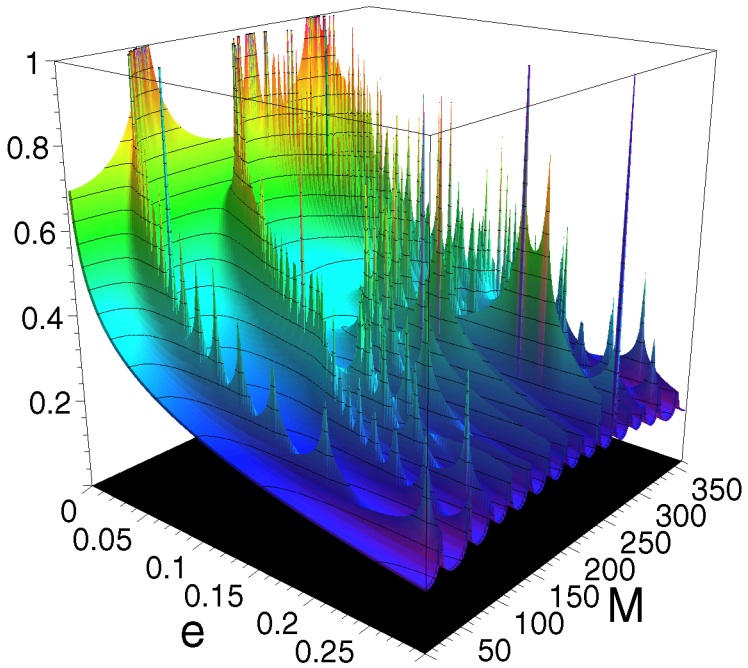
p75

μ(7,5)



p77

μ(7,7)



Postscript and GIF Output

with(*plotting*)

```
gifplot(p31, "p31", "d:/dynamics/TimeSeriesOpt/")
psplot(p31, "p31", "d:/dynamics/TimeSeriesOpt/")
gifplot(p32, "p32", "d:/dynamics/TimeSeriesOpt/")
psplot(p32, "p32", "d:/dynamics/TimeSeriesOpt/")
gifplot(p52, "p52", "d:/dynamics/TimeSeriesOpt/")
psplot(p52, "p52", "d:/dynamics/TimeSeriesOpt/")
gifplot(p33, "p33", "d:/dynamics/TimeSeriesOpt/")
psplot(p33, "p33", "d:/dynamics/TimeSeriesOpt/")
gifplot(p53, "p53", "d:/dynamics/TimeSeriesOpt/")
psplot(p53, "p53", "d:/dynamics/TimeSeriesOpt/")
gifplot(p73, "p73", "d:/dynamics/TimeSeriesOpt/")
psplot(p73, "p73", "d:/dynamics/TimeSeriesOpt/")
gifplot(p35, "p35", "d:/dynamics/TimeSeriesOpt/")
psplot(p35, "p35", "d:/dynamics/TimeSeriesOpt/")
gifplot(p55, "p55", "d:/dynamics/TimeSeriesOpt/")
psplot(p55, "p55", "d:/dynamics/TimeSeriesOpt/")
gifplot(p75, "p75", "d:/dynamics/TimeSeriesOpt/")
psplot(p75, "p75", "d:/dynamics/TimeSeriesOpt/")
gifplot(p77, "p77", "d:/dynamics/TimeSeriesOpt/")
psplot(p77, "p77", "d:/dynamics/TimeSeriesOpt/")
```

Optimization of expression size

Introduction

Suppose we encode an orbit by a series of approximations ("segments") of ϕ over timespans μ_k to orders N_k. Suppose further that there are N_{seg} timespans, and that the total span is T. Then the sum total expression "size" of the orbit, S, is proportional to the sum of the orders N_k of all the segments, $S \sim \sum_{k=1}^{N_{seg}} N_k$. If the orbit is sufficiently well-behaved, then there will be some average segment order, N_{avg}, and some average segment timespan, μ_{avg}. Hence, $N_{seg} = \frac{T}{\mu_{avg}}$ and $S \sim N_{seg} N_{avg}$, or

(1) $S \sim \frac{T N_{avg}}{\mu_{avg}}$

Now, the approximation for ϕ over a segment timespan μ is of the form $\phi = \sum_{k=0}^{N_t} \left(\sum_{n=0}^{N_e} C_{k,n}(M_0)\, e^n \right) \mu^k$, where N_t is the order of the expansion in time μ, N_e is the order of the expansion in eccentricity e, M_0 is the expansion point in time, and the $C_{n,m}$ are coefficients that are functions of M_0. The radius of convergence is one, so we require $\mu < 1$ for convergence. We can take the next term of the expansion in time as an approximate upper bound on the error due to series truncation. Suppose we have an error limit ε that we must meet for each of the segments. Then ...

Determination of the optimum time span

$S_t := \frac{a}{\mu \log(\mu)} - \frac{1}{\mu}$

$S_{t0} := \frac{a}{\mu \log(\mu)}$

$\text{factor}\left(\frac{\partial}{\partial \mu} S_{t0}\right)$

$$-\frac{a\,(\ln(\mu)+1)}{\mu^2 \ln(\mu)^2}$$

$\text{isolate}(\%, \mu)$

$$\mu = \mathbf{e}^{(-1)}$$

$mu0_approx := \text{rhs}(\%)$

$\text{collect}\left(\frac{\partial}{\partial \mu} S_t, a, factor\right)$

$$-\frac{a\,(\ln(\mu)+1)}{\mu^2 \ln(\mu)^2} + \frac{1}{\mu^2}$$

$[\text{solve}(\%, \mu)]$

$$\left[\mathbf{e}^{(1/2\,a + 1/2\sqrt{a^2+4\,a})}, \mathbf{e}^{(1/2\,a - 1/2\sqrt{a^2+4\,a})}\right]$$

$\mu 0 := \%_1$

$\text{subs}(a = \log(10^{(-12)}), \mu 0)$

$$e^{\left(-1/2\ln(1000000000000)+1/2\sqrt{\ln(1000000000000)^2-4\ln(1000000000000)}\right)}$$

```
evalf(%)
```

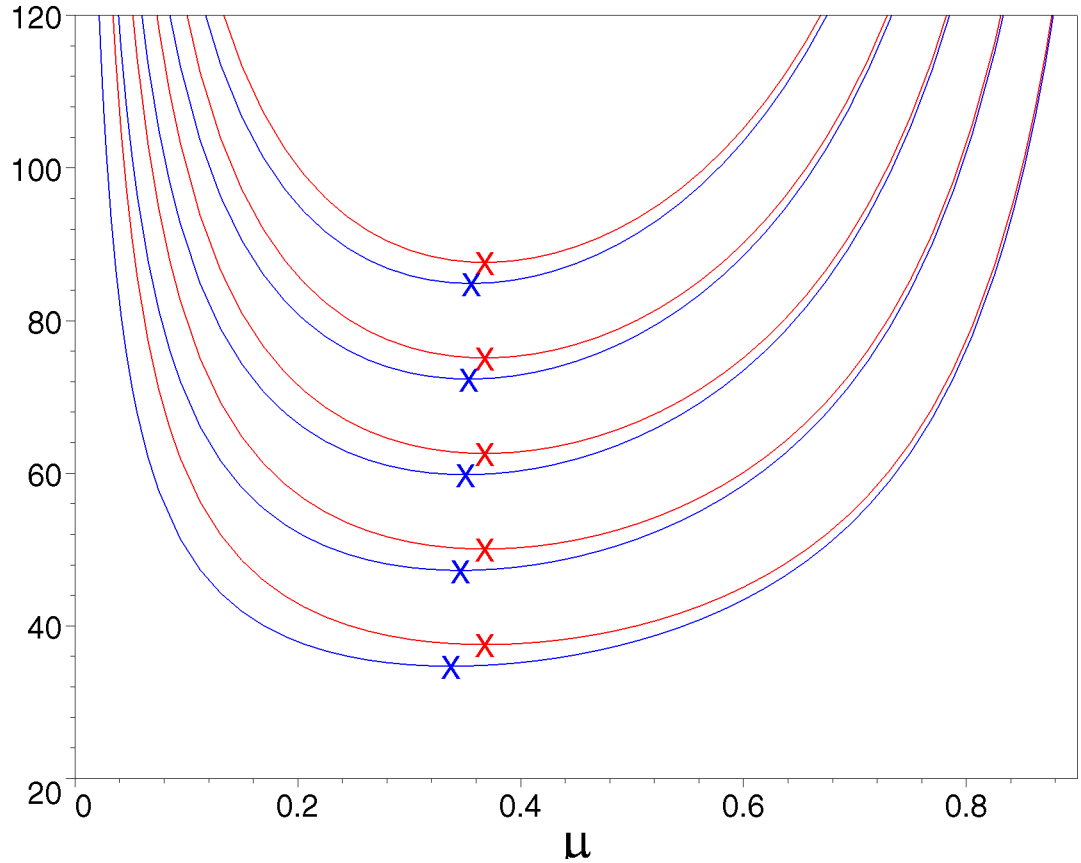
$$.3537818531$$

```
evalf(mu0_approx)
```

$$.3678794412$$

```
muplot := proc(aval)
local p, p0, pt, pt0, mumax, mumin, A;
    mumax := .9;
    mumin := .0;
    A := log(10^aval);
    p0 := plot(subs(a = A, S[t0]), μ = mumin .. mumax, color = red, thickness = 2);
    p := plot(subs(a = A, S[t]), μ = mumin .. mumax, color = blue, thickness = 2);
    pt0 := plots[textplot]([mu0_approx, subs(μ = mu0_approx, a = A, S[t0]), "X"],
        font = [HELVETICA, 12], color = red);
    pt := plots[textplot]([subs(a = A, μ0), subs(a = A, subs(μ = μ0, S[t])), "X"],
        font = [HELVETICA, 12], color = blue);
    plots[display]([p0, p, pt, pt0], view = [mumin .. mumax, 20 .. 120], labelfont = [SYMBOL, 20],
        labels = ["m", ""])
end
```

```
plots_display(muplot(−6), muplot(−8), muplot(−10), muplot(−12), muplot(−14),
labelfont = [SYMBOL, 20], labels = ["m", ""])
```



$epsvals := [10^{(-6)}, 10^{(-8)}, 10^{(-10)}, 10^{(-12)}, 10^{(-14)}]$

$Npoints := 171$

$A := \mathrm{array}(1 .. Npoints, 1 .. 2\,\mathrm{nops}(epsvals) + 1)$

$mumin := .05$

$mumax := .9$

for k **to** $Npoints$ **do**

$\quad x := mumin + \dfrac{(k - 1.0)\,(mumax - mumin)}{Npoints - 1.0};$

$\quad A_{k,1} := x;$

$\quad$ **for** j **from** 2 **to** $\mathrm{nops}(epsvals) + 1$ **do** $A_{k,j} := \mathrm{evalf}(\mathrm{subs}(a = \log(epsvals_{j-1}), \mu = x, S_{t0}))$ **od**;

$\quad$ **for** j **from** $\mathrm{nops}(epsvals) + 2$ **to** $2\,\mathrm{nops}(epsvals) + 1$ **do**

$\qquad A_{k,j} := \mathrm{evalf}(\mathrm{subs}(a = \log(epsvals_{j - \mathrm{nops}(epsvals) - 1}), \mu = x, S_t))$

$\quad$ **od**

od

writedata("d:/dynamics/TimeSeriesOpt/muplot.dat", A)

$A := \mathrm{array}(1 .. \mathrm{nops}(epsvals), 1 .. 5)$

for k **to** $\mathrm{nops}(epsvals)$ **do**

$x := epsvals_k;$

$aval := \log(x);$

$A_{k,1} := x;$

$A_{k,2} := \mathrm{evalf}(\mathrm{subs}(a = aval, mu0_approx));$

$A_{k,3} := \mathrm{evalf}(\mathrm{subs}(\mu = mu0_approx, a = aval, S_{t0}));$

$A_{k,4} := \mathrm{evalf}(\mathrm{subs}(a = aval, \mu 0));$

$A_{k,5} := \mathrm{evalf}(\mathrm{subs}(\mu = \mu 0, a = aval, S_t))$

od

writedata("d:/dynamics/TimeSeriesOpt/muzeros.dat", A)

Determination of the optimum expansion order

$$S_N := N\,\varepsilon^{\left(-\frac{1}{N+1}\right)}$$

$$S_{N0} := N\,\varepsilon^{\left(-\frac{1}{N}\right)}$$

$$\mathrm{Collect}\left(\frac{\partial}{\partial N} S_N, \varepsilon^{\left(-\frac{1}{N+1}\right)}, loc\right)$$

$$\left(1 + \frac{N \ln(\varepsilon)}{(N+1)^2}\right)\varepsilon^{\left(-\frac{1}{N+1}\right)}$$

$$[\mathrm{solve}(\%, N)]$$

$$\left[-1 - \frac{1}{2}\ln(\varepsilon) + \frac{1}{2}\sqrt{4\ln(\varepsilon) + \ln(\varepsilon)^2}, -1 - \frac{1}{2}\ln(\varepsilon) - \frac{1}{2}\sqrt{4\ln(\varepsilon) + \ln(\varepsilon)^2}\right]$$

$$N_0 := \%_1$$

$$\mathrm{Collect}\left(\frac{\partial}{\partial N} S_{N0}, \varepsilon^{\left(-\frac{1}{N}\right)}, loc\right)$$

$$\left(1 + \frac{\ln(\varepsilon)}{N}\right)\varepsilon^{\left(-\frac{1}{N}\right)}$$

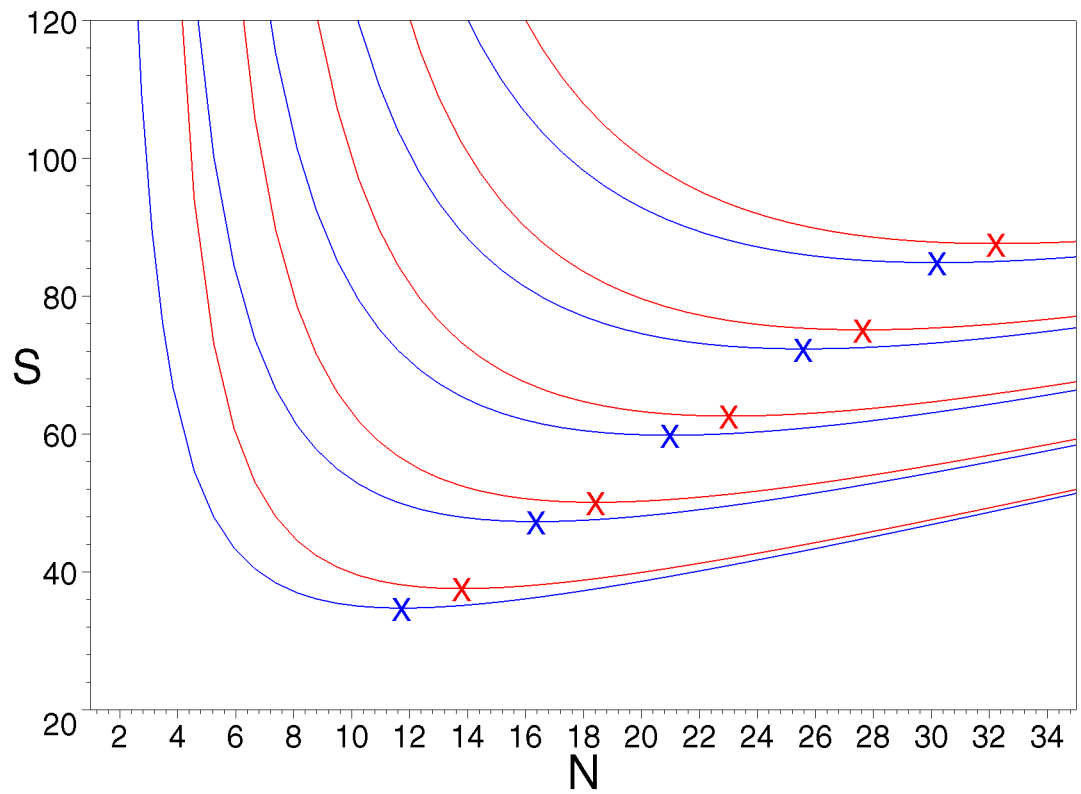
$$\mathrm{isolate}(\%, N)$$

$$N = -\ln(\varepsilon)$$

$N_0_approx := \mathrm{rhs}(\%)$

```
Nplot := proc(epsval)
local p, p0, pt, pt0, Nmax, Nmin, eps;
    Nmax := 35;
    Nmin := 1;
    eps := 10^epsval;
    p0 := plot(subs(ε = eps, S[N0]), N = Nmin .. Nmax, color = red, thickness = 2);
    p := plot(subs(ε = eps, S[N]), N = Nmin .. Nmax, color = blue, thickness = 2);
    pt0 := plots[textplot](
        [subs(ε = eps, N_0_approx), subs(N = N_0_approx, ε = eps, S[N0]), "X"],
        font = [HELVETICA, 12], color = red);
    pt := plots[textplot]([subs(ε = eps, N_0), subs(N = N_0, ε = eps, S[N]), "X"],
        font = [HELVETICA, 12], color = blue);
    plots[display]([p0, p, pt, pt0], view = [Nmin .. Nmax, 20 .. 120], labels = ["N", "S"])
end
```

$plots_{display}(\mathrm{Nplot}(-6), \mathrm{Nplot}(-8), \mathrm{Nplot}(-10), \mathrm{Nplot}(-12), \mathrm{Nplot}(-14), labels = [\text{"N"}, \text{"S"}])$



$Npoints := 171$

$epsvals := [10^{(-6)}, 10^{(-8)}, 10^{(-10)}, 10^{(-12)}, 10^{(-14)}]$

$A := \mathrm{array}(1 .. Npoints, 1 .. 11)$

$Nmin := 1$

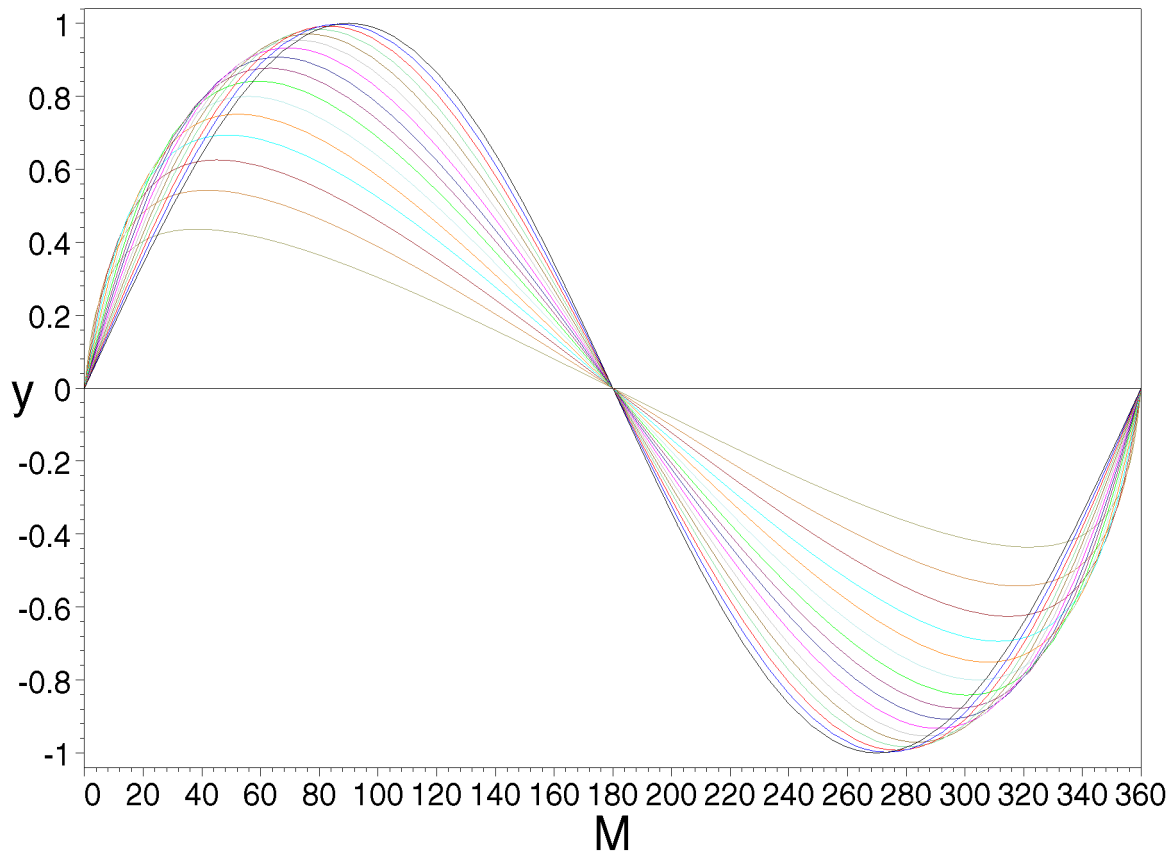
```
Nmax := 35
for k to Npoints do
    x := Nmin + (k - 1.0) (Nmax - Nmin) / (Npoints - 1.0);
    A[k, 1] := x;
    for j from 2 to nops(epsvals) + 1 do A[k, j] := evalf(subs(ε = epsvals[j - 1], N = x, S[N0])) od;
    for j from nops(epsvals) + 2 to 2 nops(epsvals) + 1 do
        A[k, j] := evalf(subs(ε = epsvals[j - nops(epsvals) - 1], N = x, S[N]))
    od
od
writedata("d:/dynamics/TimeSeriesOpt/Nplot.dat", A)
```

```
A := array(1 .. nops(epsvals), 1 .. 5)
for k to nops(epsvals) do
    x := epsvals[k];
    A[k, 1] := x;
    A[k, 2] := evalf(subs(ε = x, N_0_approx));
    A[k, 3] := evalf(subs(N = N_0_approx, ε = x, S[N0]));
    A[k, 4] := evalf(subs(ε = x, N_0));
    A[k, 5] := evalf(subs(N = N_0, ε = x, S[N]))
od
writedata("d:/dynamics/TimeSeriesOpt/Nzeros.dat", A)
```

Miscellaneous 1: y as a function of e and M

The value of the y coordinate, when the pericenter lies on the positive x axis, is $y = a\sqrt{1-e^2}\,\sin(\phi)$, where a is the semimajor axis. Here is a plot of y(e, M), with $a = 1$:

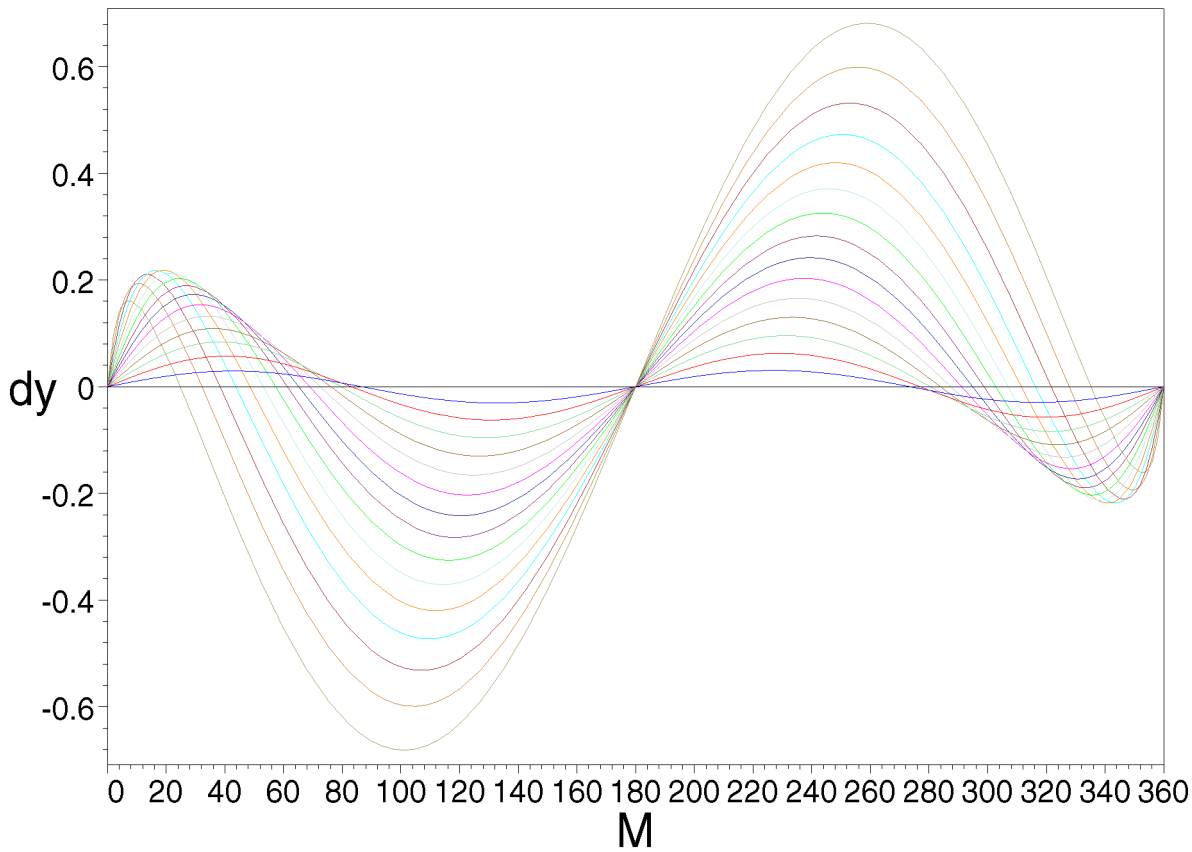
```
plots[display]([seq(plot(sqrt(1 - (.06 k)^2) sin('Keplersolve(.06 k, M deg)'), M = 0 .. 360, color = mycolors[k + 1]), k = 0 .. 15),
plot(0, M = 0 .. 360, color = black)], labels = ["M", "y"])
```



Here are plots of the difference between y(e, M) and y(0, M):

$Kdiff := (e, M) \rightarrow \sqrt{1 - e^2}\,\sin(\text{Keplersolve}(e, M)) - \sin(M)$

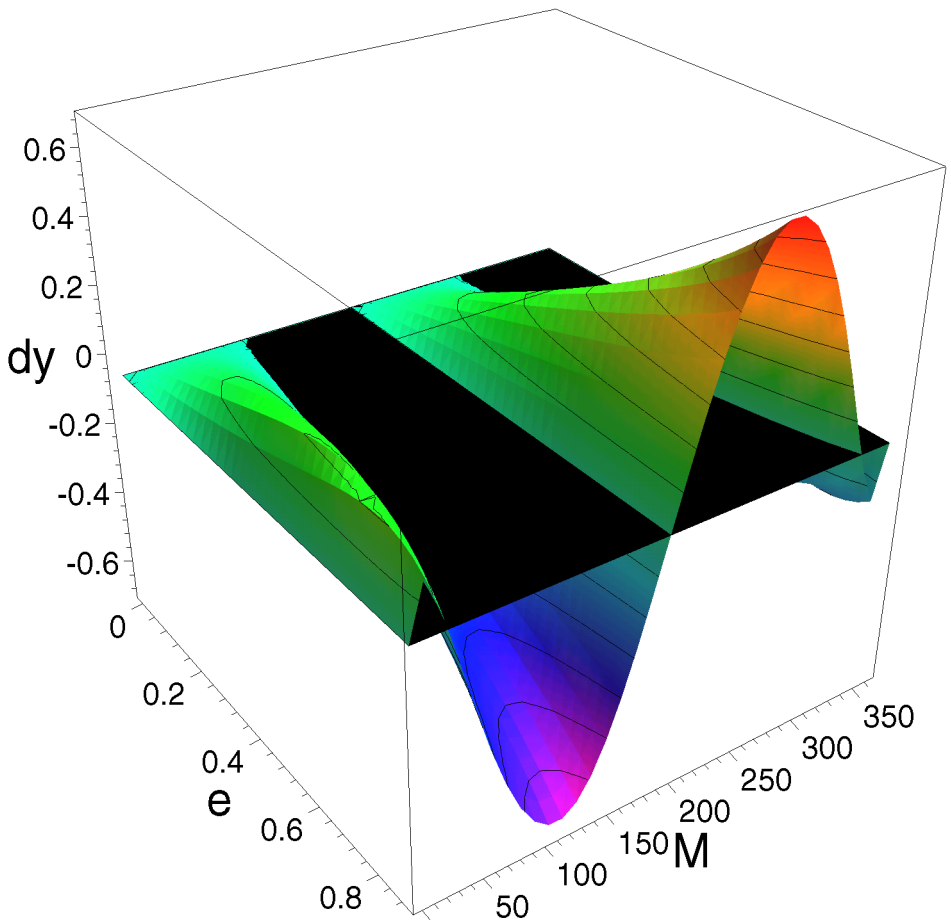
$plots_{display}([\text{seq}(\text{plot}(\text{'Kdiff}(.06\,k, M\,deg)\text{'}, M = 0 .. 360, color = mycolors_{k+1}), k = 0 .. 15)],$
$labels = [\text{"M"}, \text{"dy"}])$



$p0 := \mathrm{plot3d}(0, e = 0 .. .9, M = 0 .. 360, color = black)$

$p1 :=$
$\mathrm{plot3d}(\text{'Kdiff}(e, M\ deg)\text{'}, e = 0 .. .9, M = 0 .. 360, style = patchcontour, contours = 15, grid = [35, 35])$

$plots_{display3d}([p0, p1], orientation = [-35, 60], labels = [\text{"e"}, \text{"M"}, \text{"dy"}], lightmodel = light3)$



Miscellaneous 2: a different approximation approach

$solnd := \mathrm{remove}(has, [\mathrm{solve}(\mathrm{rhs}(ODE), \delta)], I)$

$\delta = \mathrm{op}(solnd)$

$solnd := \%$

$$\delta = \left(3\left(\frac{\partial}{\partial\xi}f(\xi)\right)\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right) - 3\,f(\xi)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)^2 - \left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^3 + \mathrm{sqrt}\left(8\left(\frac{\partial}{\partial\xi}f(\xi)\right)^3\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right) - 3\left(\frac{\partial}{\partial\xi}f(\xi)\right)^2\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^2 - 18\left(\frac{\partial}{\partial\xi}f(\xi)\right)\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)f(\xi) + 9\,f(\xi)^2\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)^2 + 6\,f(\xi)\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^3\right)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)\right)^{(1/3)} \Big/ \frac{\partial^3}{\partial\xi^3}f(\xi) - \Big($$

$$2\left(\frac{\partial}{\partial\xi}f(\xi)\right)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)-\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^2\Bigg)\Bigg/\Bigg(\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)\Bigg(3\left(\frac{\partial}{\partial\xi}f(\xi)\right)\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)$$

$$-3\,f(\xi)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)^2-\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^3+\mathrm{sqrt}\Bigg(8\left(\frac{\partial}{\partial\xi}f(\xi)\right)^3\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)-3\left(\frac{\partial}{\partial\xi}f(\xi)\right)^2\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^2$$

$$-18\left(\frac{\partial}{\partial\xi}f(\xi)\right)\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)f(\xi)+9\,f(\xi)^2\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)^2+6\,f(\xi)\left(\frac{\partial^2}{\partial\xi^2}f(\xi)\right)^3\Bigg)\left(\frac{\partial^3}{\partial\xi^3}f(\xi)\right)$$

$$\Bigg)^{(1/3)}\Bigg)-\frac{\frac{\partial^2}{\partial\xi^2}f(\xi)}{\frac{\partial^3}{\partial\xi^3}f(\xi)}$$

Check the solution.

factor(subs(%, *ODE*))

$$f(\phi)=0$$

Substitute $f(\xi)=\xi-e\sin(\xi)-M$.

eval(subs(f(ξ) = Kepler(ξ, *e*, *M*), *solnd*))

$$\delta=(3\,(1-e\cos(\xi))\,e^2\sin(\xi)\cos(\xi)-3\,(\xi-e\sin(\xi)-M)\,e^2\cos(\xi)^2-e^3\sin(\xi)^3+\mathrm{sqrt}($$

$$8\,(1-e\cos(\xi))^3\,e\cos(\xi)-3\,(1-e\cos(\xi))^2\,e^2\sin(\xi)^2$$

$$-18\,(1-e\cos(\xi))\,e^2\sin(\xi)\cos(\xi)\,(\xi-e\sin(\xi)-M)+9\,(\xi-e\sin(\xi)-M)^2\,e^2\cos(\xi)^2$$

$$+6\,(\xi-e\sin(\xi)-M)\,e^3\sin(\xi)^3)\,e\cos(\xi))^{(1/3)}/(e\cos(\xi))-($$

$$2\,(1-e\cos(\xi))\,e\cos(\xi)-e^2\sin(\xi)^2)\Big/\Big(e\cos(\xi)\,(3\,(1-e\cos(\xi))\,e^2\sin(\xi)\cos(\xi)$$

$$-3\,(\xi-e\sin(\xi)-M)\,e^2\cos(\xi)^2-e^3\sin(\xi)^3+\mathrm{sqrt}(8\,(1-e\cos(\xi))^3\,e\cos(\xi)$$

$$-3\,(1-e\cos(\xi))^2\,e^2\sin(\xi)^2-18\,(1-e\cos(\xi))\,e^2\sin(\xi)\cos(\xi)\,(\xi-e\sin(\xi)-M)$$

$$+9\,(\xi-e\sin(\xi)-M)^2\,e^2\cos(\xi)^2+6\,(\xi-e\sin(\xi)-M)\,e^3\sin(\xi)^3)\,e\cos(\xi))^{(1/3)}\Big)-\frac{\sin(\xi)}{\cos(\xi)}$$

subs(sin(ξ) = *S*, cos(ξ) = *C*, %)

$$\delta=(3\,(1-e\,C)\,e^2\,S\,C-3\,(\xi-e\,S-M)\,e^2\,C^2-e^3\,S^3+\mathrm{sqrt}(8\,(1-e\,C)^3\,e\,C-3\,(1-e\,C)^2\,e^2\,S^2$$

$$-18\,(1-e\,C)\,e^2\,S\,C\,(\xi-e\,S-M)+9\,(\xi-e\,S-M)^2\,e^2\,C^2+6\,(\xi-e\,S-M)\,e^3\,S^3)\,e\,C)^{(1/3)}/$$

$$(e\,C) - (2\,(1-e\,C)\,e\,C - e^2\,S^2) \Big/ \Big(e\,C\,(3\,(1-e\,C)\,e^2\,S\,C - 3\,(\xi - e\,S - M)\,e^2\,C^2 - e^3\,S^3 +$$

$$\mathrm{sqrt}(8\,(1-e\,C)^3\,e\,C - 3\,(1-e\,C)^2\,e^2\,S^2 - 18\,(1-e\,C)\,e^2\,S\,C\,(\xi - e\,S - M)$$

$$+ 9\,(\xi - e\,S - M)^2\,e^2\,C^2 + 6\,(\xi - e\,S - M)\,e^3\,S^3)\,e\,C)^{(1/3)}\Big) - \frac{S}{C}$$

Suppose the eccentricity is small. Then

expansion(%, *e*, 1)

$$\delta = \frac{1}{6}\,\frac{2^{(1/6)}\,(3\,S\,C - 3\,(\xi - M)\,C^2)\,e}{(\sqrt{2}\,\sqrt{e\,C}\,e\,C)^{(1/3)}\,\sqrt{e\,C}\,C}$$

$$+ \frac{1}{12}\,\frac{2^{(5/6)}\,(\sqrt{2}\,\sqrt{e\,C}\,e\,C)^{(1/3)}\,(3\,S\,C - 3\,(\xi - M)\,C^2)}{C^2\,\sqrt{e\,C}} - \frac{S}{C} + \frac{2^{(1/3)}\,(\sqrt{2}\,\sqrt{e\,C}\,e\,C)^{(1/3)}}{e\,C}$$

$$- \frac{2^{(2/3)}}{(\sqrt{2}\,\sqrt{e\,C}\,e\,C)^{(1/3)}}$$

Not good.

read/save

save "d:/dynamics/TimeSeriesOpt/Kepler.m"

read "d:/dynamics/TimeSeriesOpt/Kepler.m"

save
p31, *p32*, *p52*, *p33*, *p53*, *p73*, *p35*, *p55*, *p75*, *p77*, "d:/dynamics/TimeSeriesOpt/timespan_3D_plots.m"

read "d:/dynamics/TimeSeriesOpt/timespan_3D_plots.m"

?