Intersections with a Sphere

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A Plane and a Sphere

Define a plane and a sphere:

restart

$eq1 := p + q + r = 1$

$eq2 := p^2 + q^2 + r^2 = 1$

Plot them, along with the projections of their intersection.

with(*plots*)

with(*plottools*)

$P :=$ implicitplot3d($eq1$, $p = -1.2 .. 1.2$, $q = -1.2 .. 1.2$, $r = -1.2 .. 1.2$, $style = patchnogrid$, $color = gold$)

$S :=$ implicitplot3d($eq2$, $p = -1.2 .. 1.2$, $q = -1.2 .. 1.2$, $r = -1.2 .. 1.2$, $style = patchcontour$, $grid = [20, 20, 20]$, $color = blue$)

$varlist := [p, q, r]$

for j **in** $varlist$ **do**

$eq_.j :=$ op(eliminate($\{eq1, eq2\}, j$)$_2$);

$x :=$ remove($has, varlist, j$)$_1$;

$y :=$ remove($has, varlist, j$)$_2$;

$C_.j :=$ implicitplot($eq_.j$, $x = -1.2 .. 1.2$, $y = -1.2 .. 1.2$, $color = red$, $thickness = 2$)

od

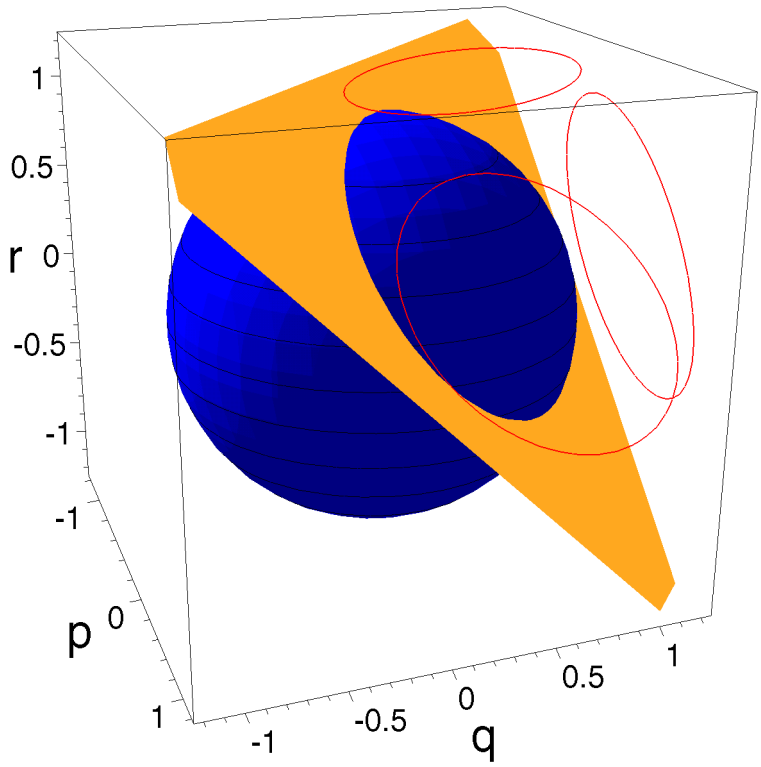
$f_p :=$ transform($(x, y) \rightarrow [1.2, x, y]$)

$f_q :=$ transform($(x, y) \rightarrow [x, 1.2, y]$)

$f_r :=$ transform($(x, y) \rightarrow [x, y, 1.2]$)

display(P, S, seq($f_$.(op(i, $varlist$))($C_$.(op(i, $varlist$))), $i = 1 .. 3$), $orientation = [-20, 70]$,

labels = ["p", "q", "r"], *scaling* = *constrained*, *lightmodel* = *light2*)



The Projection of the Intersection is an Ellipse

One Ellipse

The explicit solutions for (p,q), parametrized by r, of the intersection curve are

[allvalues(solve({ *eq1*, *eq2* }, { *p*, *q* }))]

$$\left[\{q=-\frac{1}{2}r+\frac{1}{2}+\frac{1}{2}\sqrt{-3r^2+2r+1},\ p=-\frac{1}{2}r+\frac{1}{2}-\frac{1}{2}\sqrt{-3r^2+2r+1}\},\ \{q=-\frac{1}{2}r+\frac{1}{2}-\frac{1}{2}\sqrt{-3r^2+2r+1},\ p=-\frac{1}{2}r+\frac{1}{2}+\frac{1}{2}\sqrt{-3r^2+2r+1}\}\right]$$

p and q are interchangable. Plot both sets:

sols1 := [rhs(op(select(*has*, $\%_1$, *p*))), rhs(op(select(*has*, $\%_1$, *q*)))]

sols2 := [rhs(op(select(*has*, $\%\%_2$, *p*))), rhs(op(select(*has*, $\%\%_2$, *q*)))]

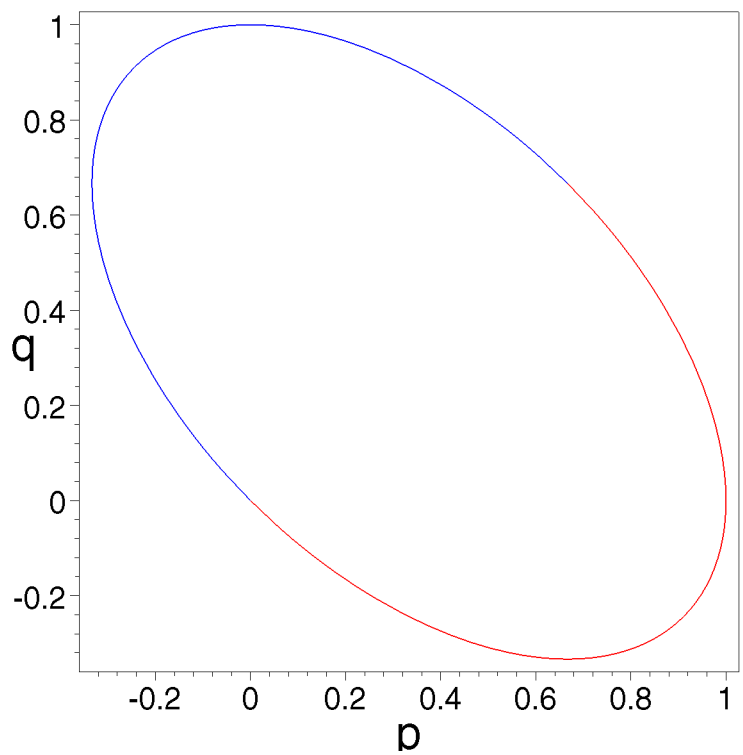
$$sols1 := \left[-\frac{1}{2}r+\frac{1}{2}-\frac{1}{2}\sqrt{-3r^2+2r+1}, -\frac{1}{2}r+\frac{1}{2}+\frac{1}{2}\sqrt{-3r^2+2r+1}\right]$$

$$sols2 := \left[-\frac{1}{2}r+\frac{1}{2}+\frac{1}{2}\sqrt{-3r^2+2r+1}, -\frac{1}{2}r+\frac{1}{2}-\frac{1}{2}\sqrt{-3r^2+2r+1}\right]$$

$$\left[\mathrm{solve}\left(\mathrm{select}(has, sols1_1, r^2)^2, r\right)\right]$$

$$\left[\frac{-1}{3}, 1\right]$$

plot([[op(*sols1*), $r = \%_1 .. \%_2$], [op(*sols2*), $r = \%_1 .. \%_2$]], *scaling* = *constrained*, *color* = [*blue*, *red*], *thickness* = 2, *labels* = ["p", "q"])



This is, of course, just the ellipse that is the plane-sphere intersection projected onto the (p,q) plane.

Another Ellipse

Let's look at the intersection parametrized by (p,q).

eliminate({*eq1*, *eq2*}, *r*)

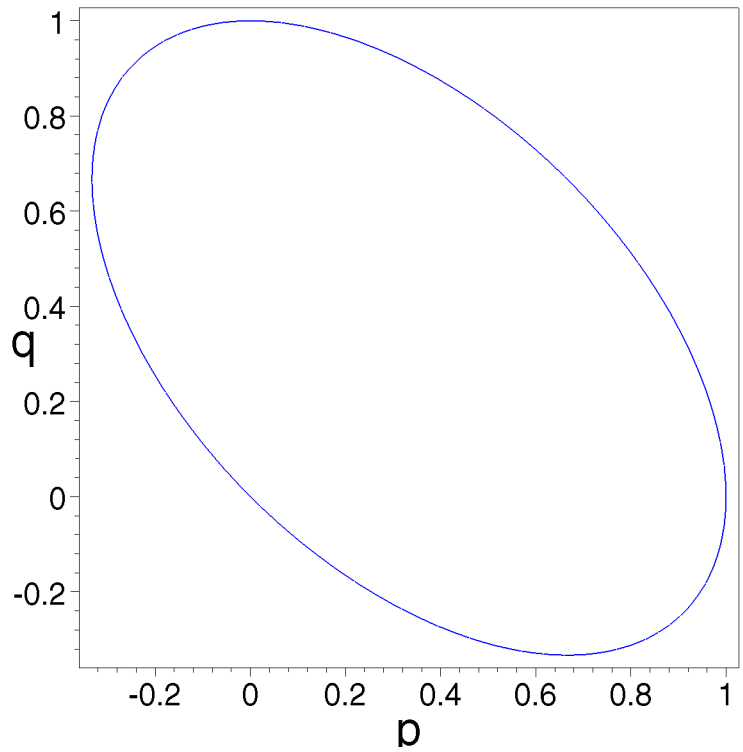
$$[\{r = -p-q+1\}, \{p^2+q^2+pq-p-q\}]$$

$pq := \mathrm{op}(\%_2) = 0$

$$pq := p^2 + q^2 + p\,q - p - q = 0$$

which is an equation for an ellipse. Plotting this equation implicitly produces the same ellipse as above.

implicitplot(pq, $p = -1 .. 1$, $q = -1 .. 1$, *grid* = [50, 50], *scaling* = *constrained*, *color* = *blue*, *thickness* = 2)



Ellipses the Pun ...

We can remove the cross term with a rotation:

$pqsubs := [p = u\cos(\theta) - v\sin(\theta),\ q = u\sin(\theta) + v\cos(\theta)]$

subs($pqsubs$, pq)

collect(%, [u, v], *combine*, *distributed*)

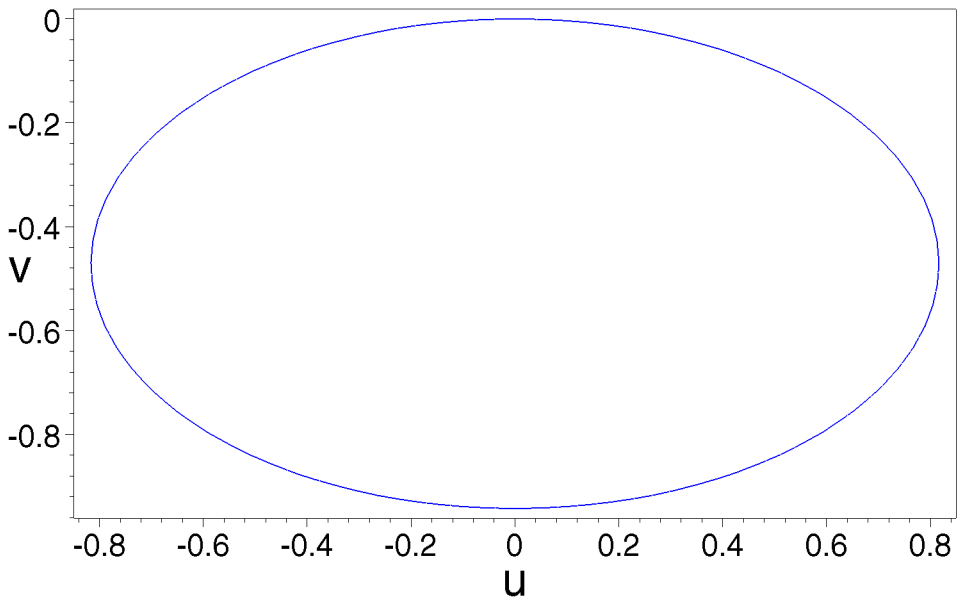
$$\left(1 + \frac{1}{2}\sin(2\,\theta)\right)u^2 + \cos(2\,\theta)\,v\,u + (-\cos(\theta) - \sin(\theta))\,u + (\sin(\theta) - \cos(\theta))\,v + \left(-\frac{1}{2}\sin(2\,\theta) + 1\right)v^2 = 0$$

$\mathrm{eval}\left(\mathrm{subs}\left(\theta = \frac{3\,\pi}{4}, \%\right)\right)$

$$\frac{1}{2}u^2 + \sqrt{2}\,v + \frac{3}{2}v^2 = 0$$

$uv := \%$

implicitplot(uv, $u = -1 .. 1$, $v = -1 .. 1$, $grid = [50, 50]$, $scaling = constrained$, $color = blue$, $thickness = 2$)



We can easily put the equation into standard form:

csquare(uv, v)

$$\frac{3}{2}\left(v+\frac{1}{3}\sqrt{2}\right)^2+\frac{1}{2}u^2-\frac{1}{3}=0$$

map($x \rightarrow 3x+1$, %)

$$\frac{9}{2}\left(v+\frac{1}{3}\sqrt{2}\right)^2+\frac{3}{2}u^2=1$$

An Interesting Surface

Recall the sphere and plane equations:

eq1

eq2

$$p+q+r=1$$

$$p^2+q^2+r^2=1$$

Now, consider map($x \to x^2$, *eq1*)

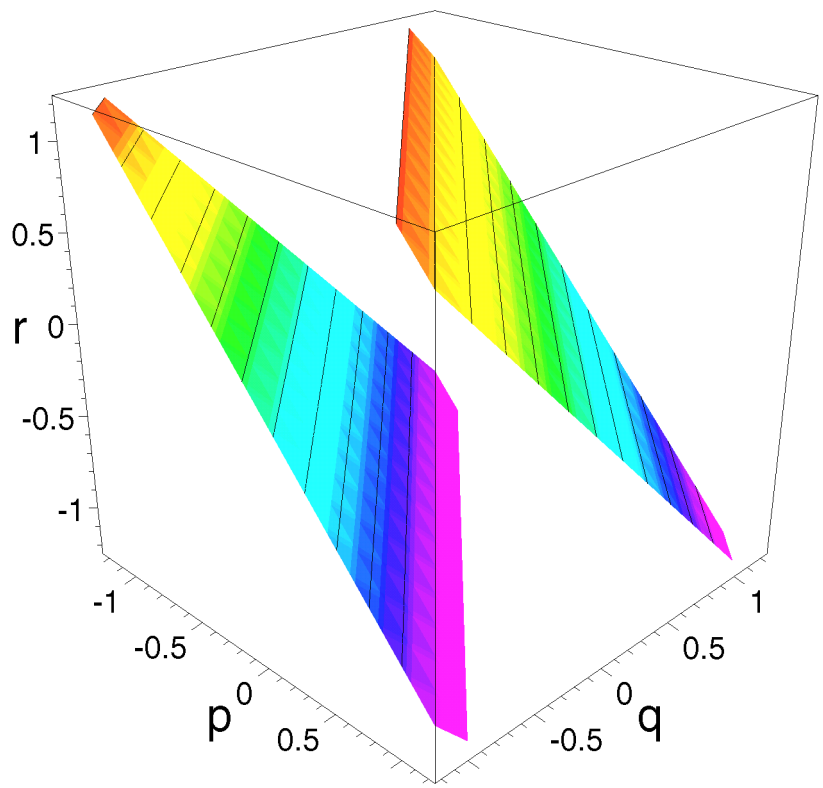
$$(p+q+r)^2 = 1$$

eqQ := %

This is a quadratic surface whose appearance is

Q := implicitplot3d(*eqQ*, $p = -1.2 .. 1.2$, $q = -1.2 .. 1.2$, $r = -1.2 .. 1.2$, *style* = *patchcontour*, *grid* = [15, 15, 15])

display(*Q*, *labels* = ["p", "q", "r"], *orientation* = [−45, 65], *scaling* = *constrained*)



We've doubled our plane. Substitute the sphere equation into this to get

simplify(*eqQ*, {*eq2*})

$$2\,p\,q + 2\,p\,r + 2\,q\,r + 1 = 1$$

$\text{map}\left(x \to \frac{x-1}{2}, \%\right)$

$$p\,q + p\,r + q\,r = 0$$

eq3 := %

Another way to view this is isolate(*eq3*, *r*)

$$r = -\frac{p\,q}{p+q}$$

$eq_r := \mathrm{rhs}(\%)$

So we have a hyperboloid. A plot of this equation is

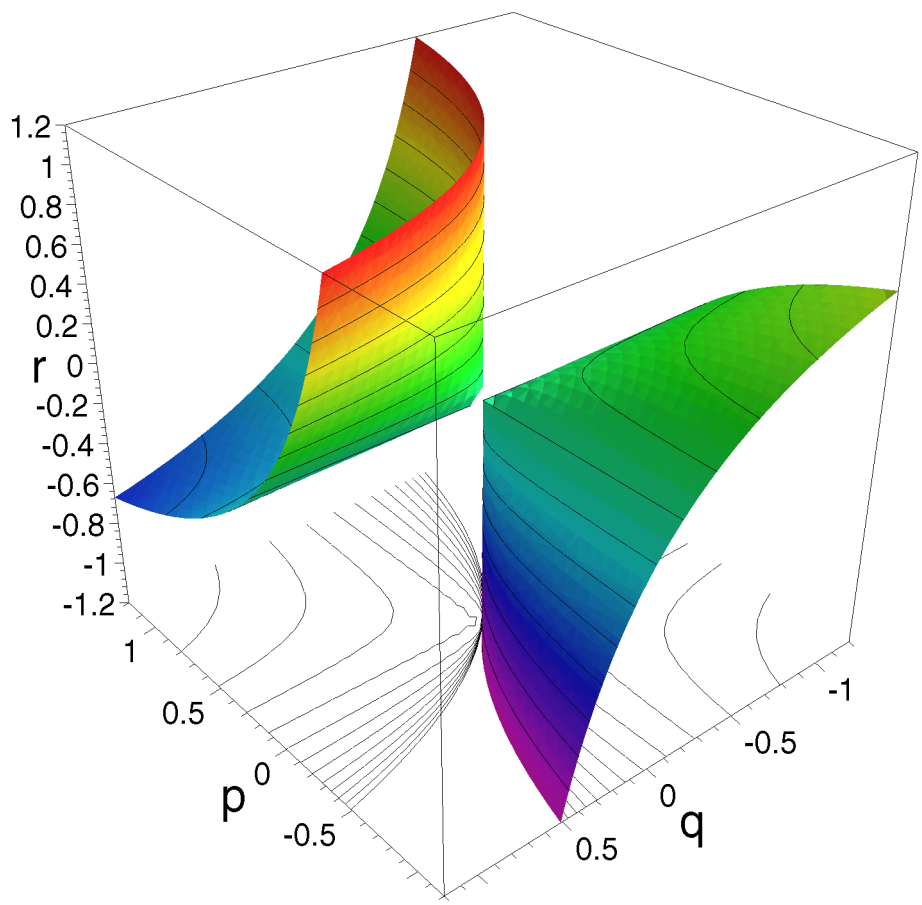
$$rvals := \left[\mathrm{seq}\left(-1.2 + \frac{i\,2.4}{16}, i = 1 .. 16\right)\right]$$

for i **to** 16 **do** $c.i := \mathrm{implicitplot}(\mathrm{subs}(r = \mathrm{op}(i, rvals), eq3), p = -1.2 .. 1.2, q = -1.2 .. 1.2,$
$scaling = constrained, color = black)$
od

$f := \mathrm{transform}((x, y) \rightarrow [x, y, -1.2])$

$H := \mathrm{implicitplot3d}(eq3, p = -1.2 .. 1.2, q = -1.2 .. 1.2, r = -1.2 .. 1.2, style = patchcontour,$
$grid = [35, 35, 35], contours = rvals)$

$\mathrm{display}(\mathrm{seq}(\mathrm{f}(c.i), i = 1 .. 16), H, labels = [\text{"p"}, \text{"q"}, \text{"r"}], orientation = [140, 60],$
$view = [-1.2 .. 1.2, -1.2 .. 1.2, -1.2 .. 1.2])$



Let's put this in standard form. First, rotate in the (q,r) plane.

$qrsubs := [q = u\cos(\theta) - v\sin(\theta), r = u\sin(\theta) + v\cos(\theta)]$

subs($qrsubs$, $eq3$)

collect(%, [u, v], *combine*, *distributed*)

$$\frac{1}{2}\sin(2\theta)u^2 + \cos(2\theta)vu + (p\cos(\theta) + p\sin(\theta))u + (-p\sin(\theta) + p\cos(\theta))v - \frac{1}{2}\sin(2\theta)v^2 = 0$$

$$\mathrm{eval}\left(\mathrm{subs}\left(\theta = \frac{3\pi}{4}, \%2\right)\right)$$

$$-u^2 - 2p\sqrt{2}v + v^2 = 0$$

Next, rotate in the (p,v) plane.

$pvsubs := [p = s\cos(\theta) - t\sin(\theta), v = s\sin(\theta) + t\cos(\theta)]$

collect(subs($pvsubs$, %%), [s, t], *combine*, *distributed*)

$$\left(-\sqrt{2}\sin(2\theta) + \frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)s^2 + (-2\sqrt{2}\cos(2\theta) + \sin(2\theta))ts + \left(\sqrt{2}\sin(2\theta) + \frac{1}{2}\cos(2\theta) + \frac{1}{2}\right)t^2 - u^2 = 0$$

isolate(remove(has, lhs(%), {s^2, t^2, u^2}), θ)

$$\theta = \frac{1}{2}\arctan(2\sqrt{2})$$

eval(subs(%, %%))

$$-s^2 + 2t^2 - u^2 = 0$$

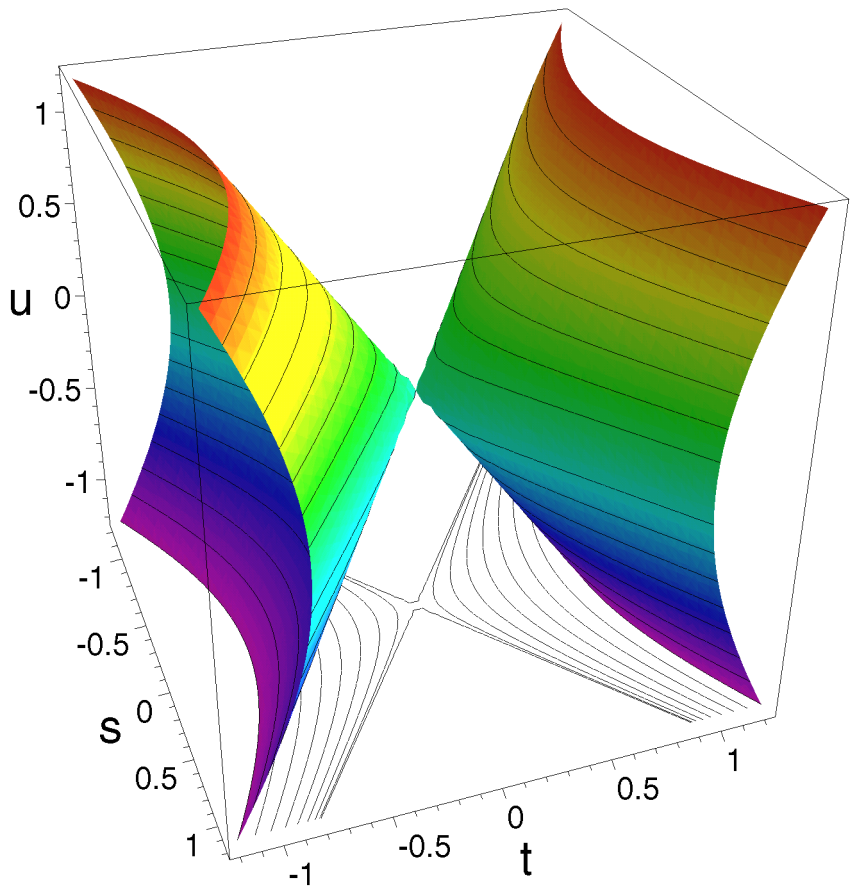
stu := %

Much better. Plot this:

stu_plot := implicitplot3d(stu, $s = -1.2 .. 1.2$, $t = -1.2 .. 1.2$, $u = -1.2 .. 1.2$, $style = patchcontour$, $grid = [35, 35, 35]$, $contours = rvals$)

for i **to** 16 **do** $c.i$:= implicitplot(subs(u = op(i, $rvals$), stu), $s = -1.2 .. 1.2$, $t = -1.2 .. 1.2$, $scaling = constrained$, $color = black$, $grid = [50, 50]$)
od

display(seq(f($c.i$), $i = 1 .. 16$), stu_plot, $labels$ = ["s", "t", "u"], $orientation = [-20, 60]$, $scaling = constrained$)

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read "c:/Maple/SpherePlane.m"
save "c:/Maple/SpherePlane.m"
?
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