Plane Curves

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restart

with(*curves*)

[*ParametrizedToImplicit*, *arclength*, *astroid*, *binormal*, *cardioid*, *cassinian*, *catenary*, *circle*, *cissoid*, *clotharg*, *clothoid*, *curvature*, *curveinverse*, *cycloid*, *deltoid*, *ellipse*, *evolute*, *folium*, *helix*, *hyperbola*, *involute*, *lemniscate*, *lissajous*, *logspiral*, *parabola*, *radial*, *tangent*, *torsion*, *torusknot*, *tractrix*, *twicubic*, *unitnormal*]

Color Scheme for 2D Plots

```
black        the curve
sienna       radial
red          evolute
blue         involute
magenta      pedal
aquamarine   inversion
```

Conics

Circle

circle = mat(circle(*a*, *t*))

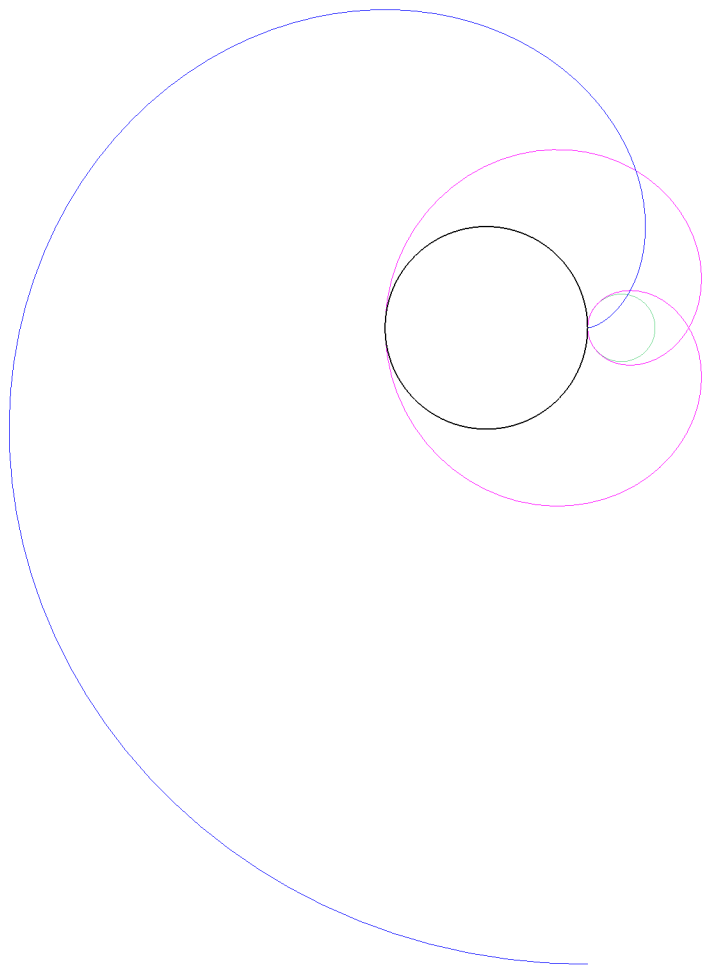
$$circle = \begin{bmatrix} a\cos(t) \\ a\sin(t) \end{bmatrix}$$

ParametrizedToImplicit(circle(*a*, *t*), *t*, [*x*, *y*])

$$y^2 + x^2 - a^2 = 0$$

curveplot2D(circle(1, *t*), *t* = 0 .. evalf(2 π), *evolute*, *involute*, *radial*, *pedal*, *inversion*, *radius* = 1, *refpoint* = [2, 0], *title* = "Circle", *axes* = *none*)

Circle



Curvature

$\kappa_{circle} = \text{simplify}(\text{curvature}(\text{circle}(a, t), t), assume = positive)$

$$\kappa_{circle} = \frac{1}{a}$$

Ellipse

$ellipse = \text{mat}(\text{ellipse}(a, b, t))$

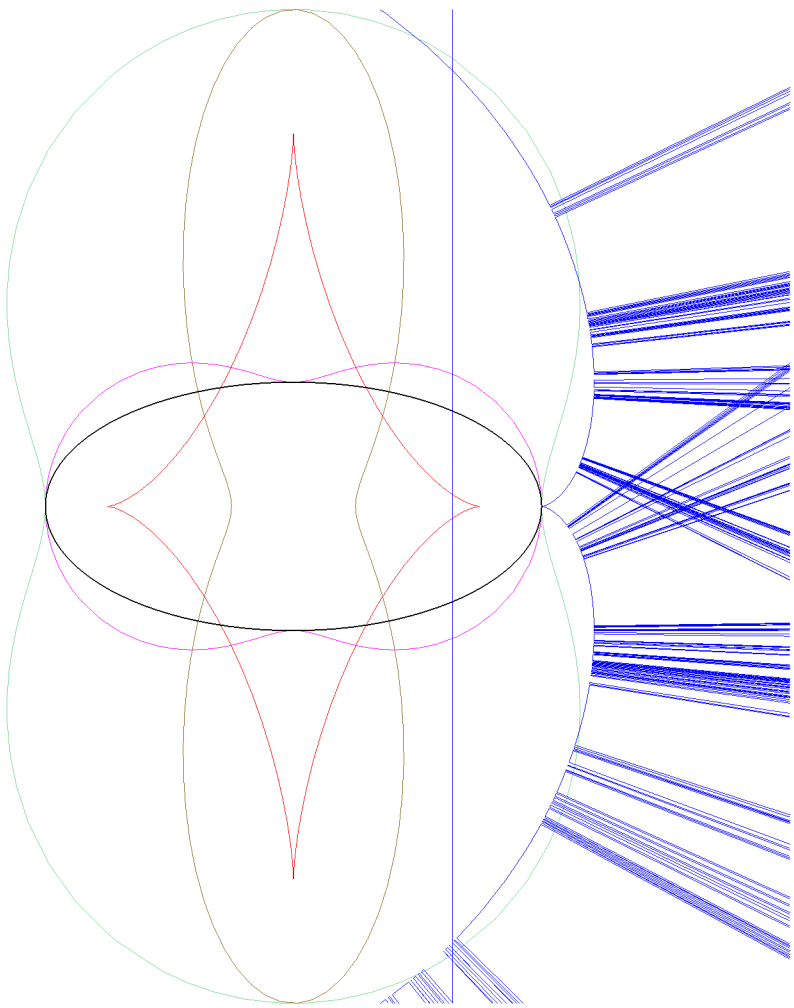
$$ellipse = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix}$$

$$\text{expand}\left(\frac{\text{ParametrizedToImplicit}(\text{ellipse}(a, b, t), t, [x, y])}{(a\,b)^2}\right)$$

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} - 1 = 0$$

curveplot2D(ellipse(2, 1, t), $t = 0$ .. evalf(2 π), *evolute*, *involute*, *radial*, *pedal*, *inversion*, *radius* = 2, *refpoint* = [0, 0], *title* = "Ellipse", *axes* = *none*, *view* = [−4 .. 4, −4 .. 4])

Ellipse

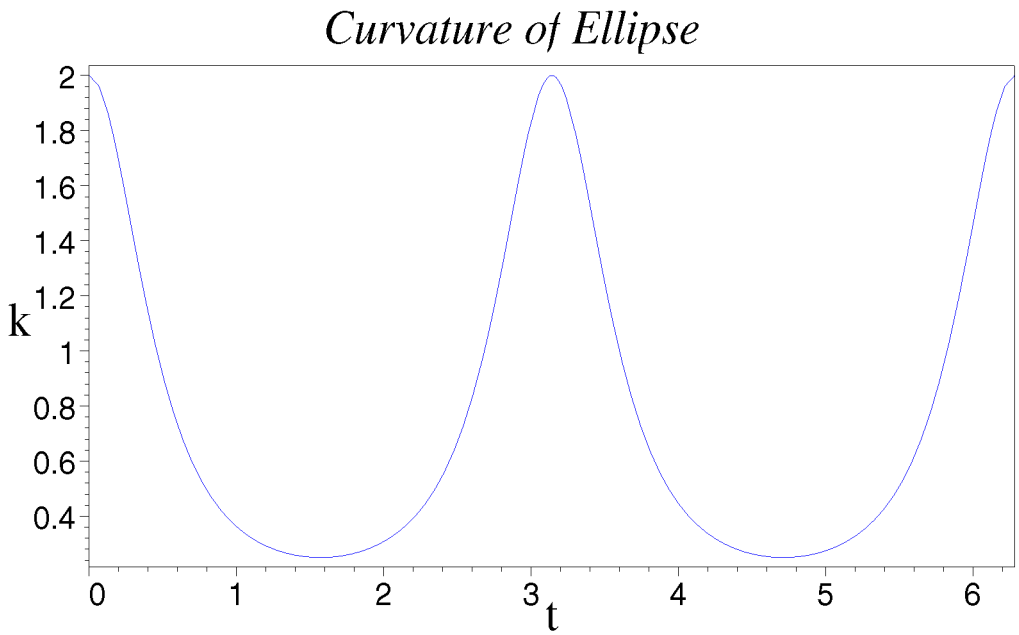


Curvature

$\kappa_{ellipse}$ = curvature(ellipse(a, b, t), t)

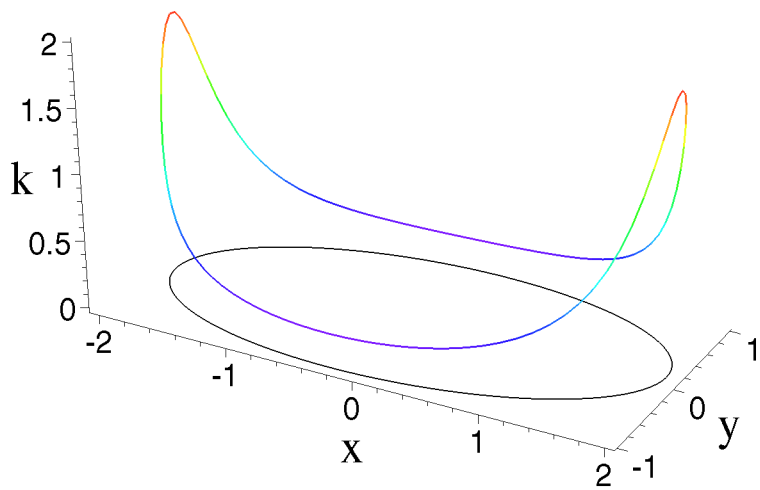
$$\kappa_{ellipse} = \frac{|a\, b|}{(a^2 \sin(t)^2 + b^2 \cos(t)^2)^{(3/2)}}$$

scalarplot(curvature(ellipse(2, 1, t), t), $t = 0$ .. 2 π, *title* = "Curvature of Ellipse", *labels* = ["t", "k"])



curvatureplot3D(ellipse(2, 1, t), curvature(ellipse(2, 1, t), t), $t = 0$ .. evalf(2 π), "Curvature of Ellipse", *axes* = *framed*, *orientation* = [−60, 70], *scaling* = *constrained*)

Curvature of Ellipse



curveplot2D(ellipse(2, 1, t), $t = 0$ .. evalf(2 π), "evolute", "radial", "pedal", "inversion", *radius* = 2, *refpoint* = [0, 0], *view* = [−4 .. 4, −4 .. 4])

Parabola

parabola = mat(parabola(f, t))

$$parabola = \begin{bmatrix} 2\,f\,t \\ f\,t^2 \end{bmatrix}$$

isolate(ParametrizedToImplicit(parabola(f, t), ['x', 'y'], t), y)

$$y = \frac{1}{4}\,\frac{x^2}{f}$$

$\kappa_{parabola}$ = simplify(kappa2D(parabola(f, t), t), *assume* = *positive*)

$$\kappa_{parabola} = \frac{1}{2} \frac{1}{f\,(1+t^2)^{(3/2)}}$$

Hyperbola

$hyperbola = \mathrm{mat}(\mathrm{hyperbola}(a, b, t))$

$$hyperbola = \begin{bmatrix} a\cosh(t) \\ b\sinh(t) \end{bmatrix}$$

$\mathrm{expand}\left(\frac{\mathrm{ParametrizedToImplicit}(\mathrm{hyperbola}(a, b, t), ['x', 'y'], t)}{(a\,b)^2}\right)$

$$1 + \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$$

$\kappa_{hyperbola} = \mathrm{simplify}(\mathrm{kappa2D}(\mathrm{hyperbola}(a, b, t), t))$

$$\kappa_{hyperbola} = \frac{|a\,b|}{(a^2\cosh(t)^2 - a^2 + b^2\cosh(t)^2)^{(3/2)}}$$

Lemniscate of Bernoulli

$lemniscate = \mathrm{mat}(\mathrm{lemniscate}(a, t))$

$$lemniscate = \begin{bmatrix} \frac{a\cos(t)}{1+\sin(t)^2} \\ \frac{a\sin(t)\cos(t)}{1+\sin(t)^2} \end{bmatrix}$$

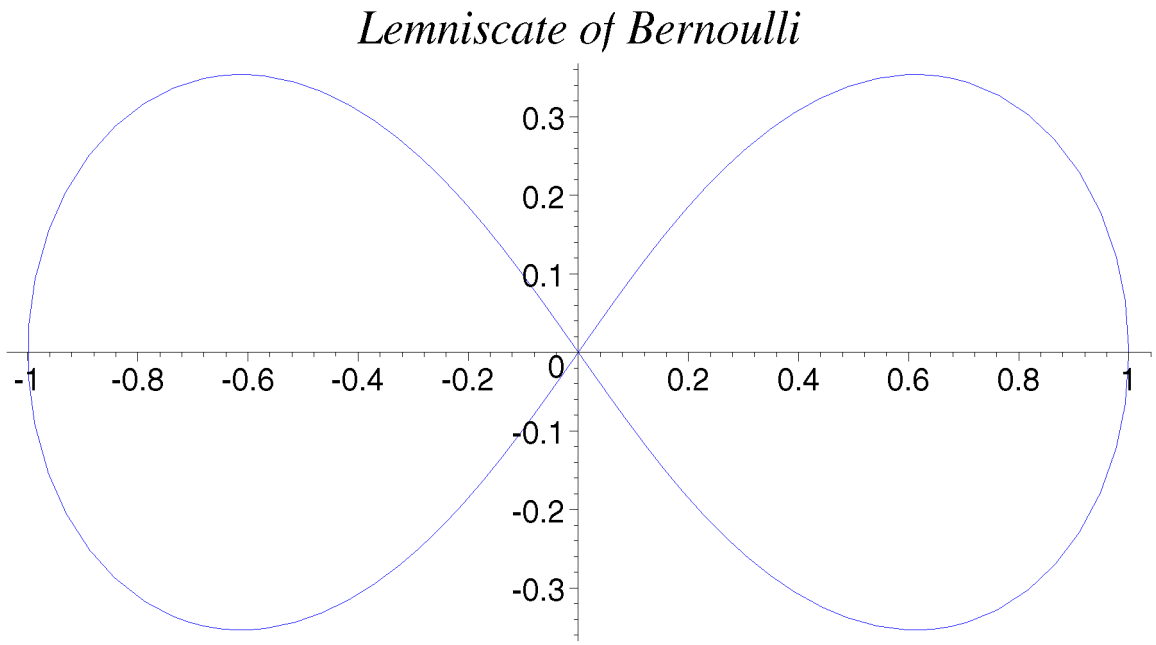
$\mathrm{ParametrizedToImplicit}(\mathrm{lemniscate}(a, t), ['x', 'y'], t)$

$\mathrm{collect}(\%, a, factor)$

$\mathrm{applyop}(collect, [1, 1], \%, [a, y])$

$$(y^2 - x^2)\,a^2 + (x^2 + y^2)^2 = 0$$

$\mathrm{curveplot2D}(\mathrm{lemniscate}(1, t), t = 0 .. 2\,\pi, axes = normal, title = \text{"Lemniscate of Bernoulli"})$



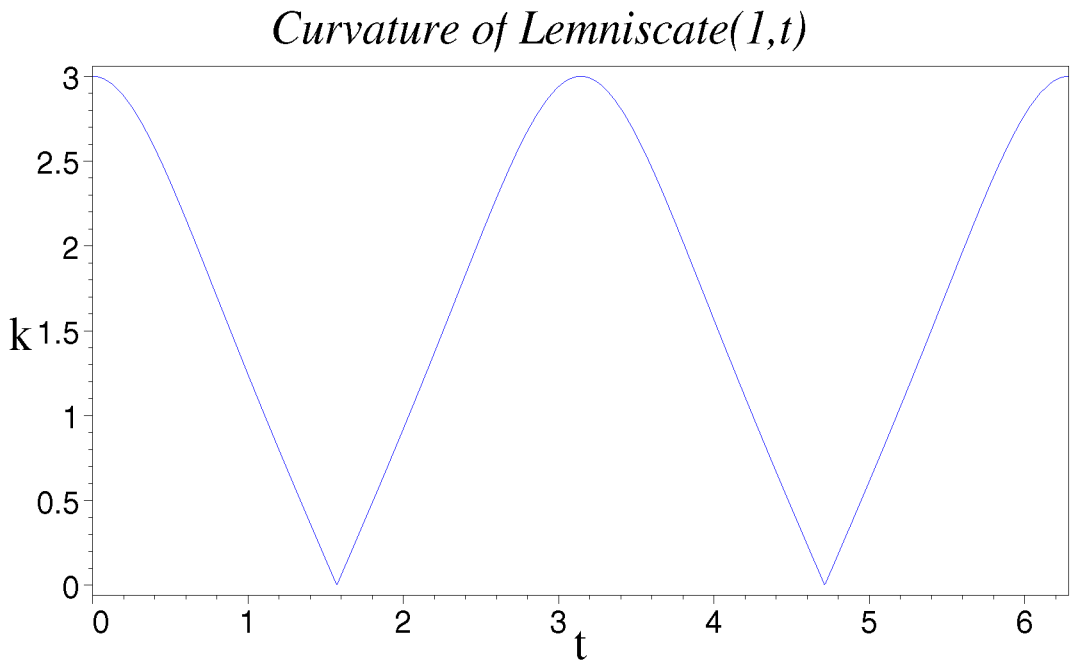
Curvature

$\kappa = \text{simplify}(\text{kappa2D}(\text{lemniscate}(a, t), t), trig)$

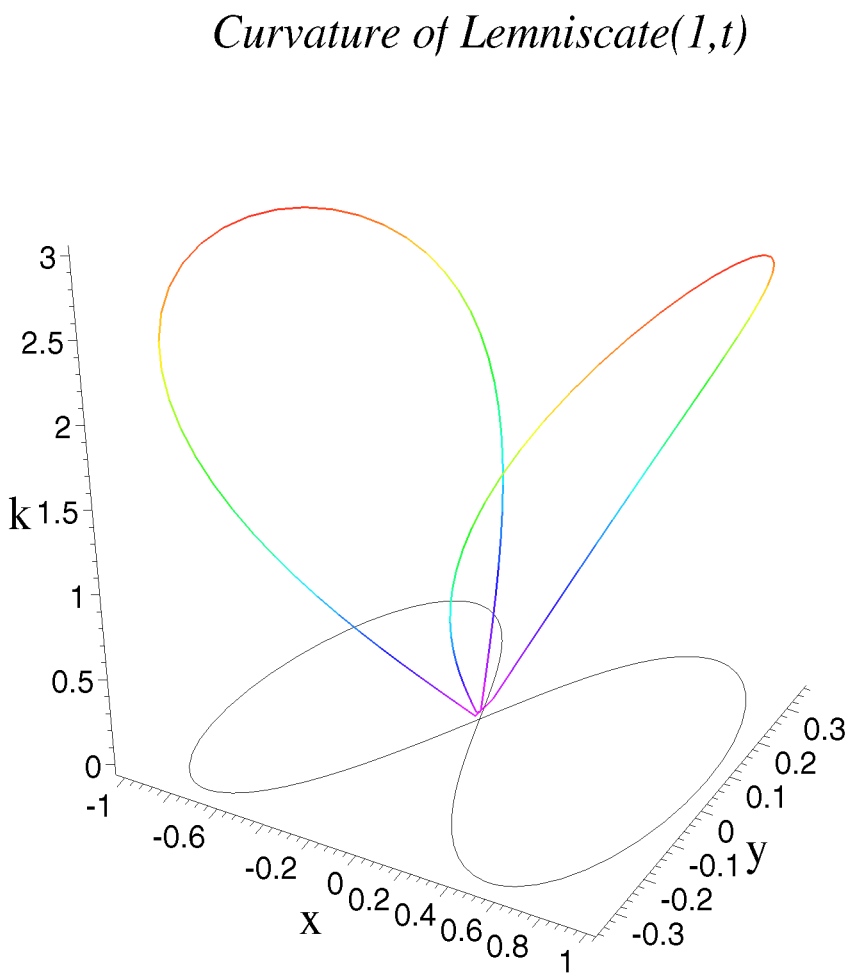
$eq := \text{fn}(\text{rhs}(\%), a, t)$

$$\kappa = 3 \frac{\left| \frac{a^2 \cos(t)}{\cos(t)^4 - 4\cos(t)^2 + 4} \right|}{\left(-\frac{a^2}{-2 + \cos(t)^2} \right)^{(3/2)}}$$

$\text{scalarplot}(\text{eq}(1, t), t = 0 .. 2\pi, title = \text{"Curvature of Lemniscate(1,t)"}, labels = [\text{"t"}, \text{"k"}])$



curvatureplot3D(lemniscate(1, *t*), eq(1, *t*), *t* = 0 .. 2 π, "Curvature of Lemniscate(1,t)", *axes* = *framed*, *orientation* = [−60, 70])



Cissoid of Diocles

$cissoid = \mathrm{mat}(\mathrm{cissoid}(a, t))$

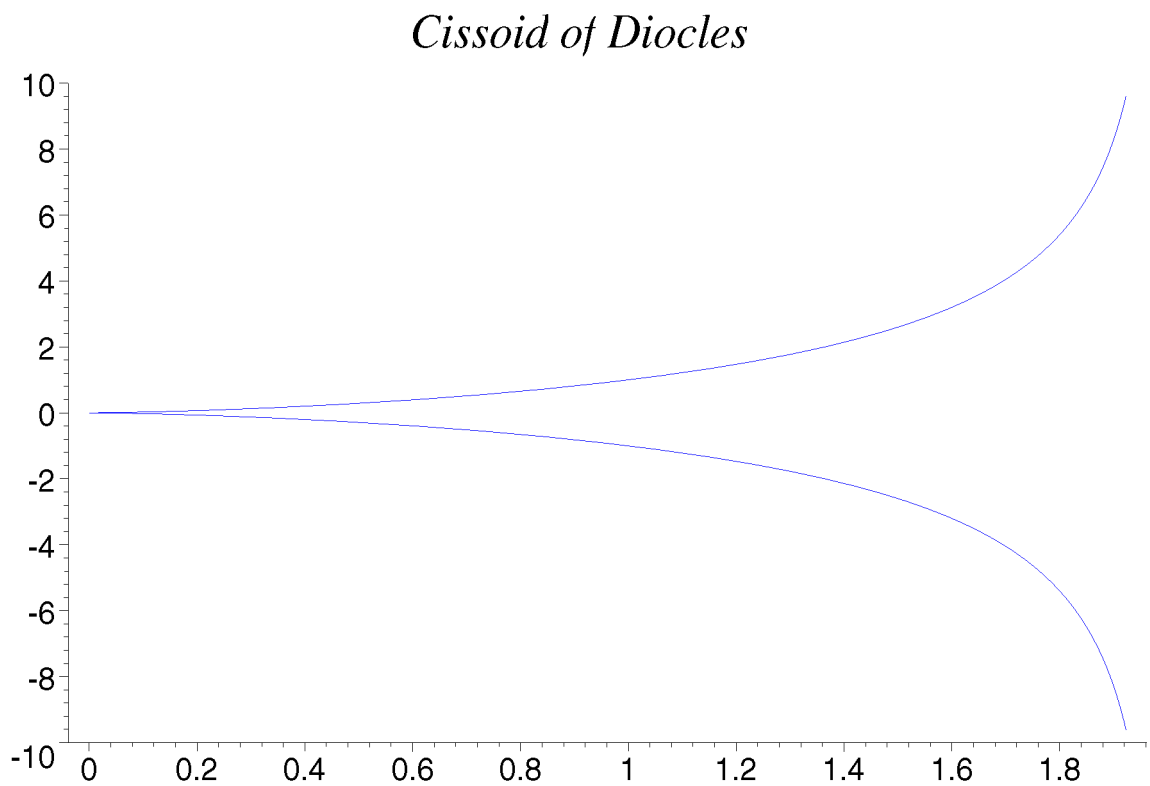
$$cissoid = \begin{bmatrix} 2\,\dfrac{a\,t^2}{1+t^2} \\ 2\,\dfrac{a\,t^3}{1+t^2} \end{bmatrix}$$

$\mathrm{ParametrizedToImplicit}(\mathrm{cissoid}(a, t), ['x', 'y'], t)$

$\mathrm{isolate}(\%, y^2)$

$$y^2 = \frac{x^3}{-x + 2\,a}$$

$\mathrm{curveplot2D}(\mathrm{cissoid}(1, t), t = -5 .. 5, title = \text{"Cissoid of Diocles"})$

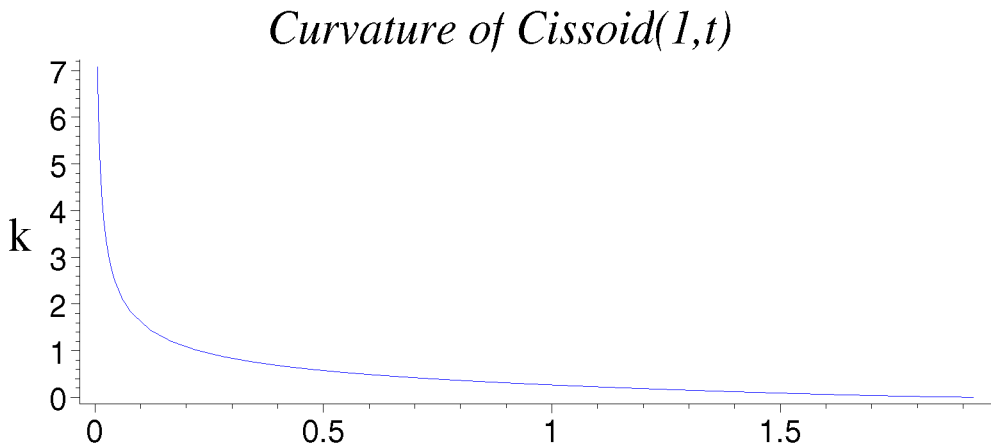


Curvature

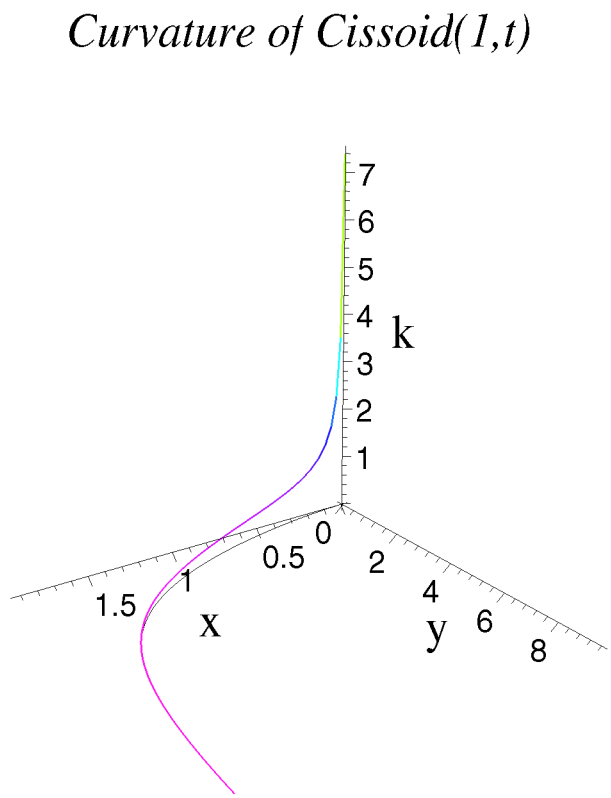
$\kappa = \mathrm{simplify}(\mathrm{kappa2D}(\mathrm{cissoid}(a, t), t), assume = real)$

$$\kappa = 3\,\frac{\mathrm{signum}(a)\,\mathrm{signum}(t)}{a\,t\,(t^2+4)^{(3/2)}}$$

curveplot2D([cissoid(1, t)$_1$, kappa2D(cissoid(1, t), t)], t = 0 .. 5, *title* = "Curvature of Cissoid(1,t)", *labels* = ["t", "k"])



curvatureplot3D(cissoid(1, t), kappa2D(cissoid(1, t), t), t = 0 .. 5, "Curvature of Cissoid(1,t)", *axes* = *normal*)

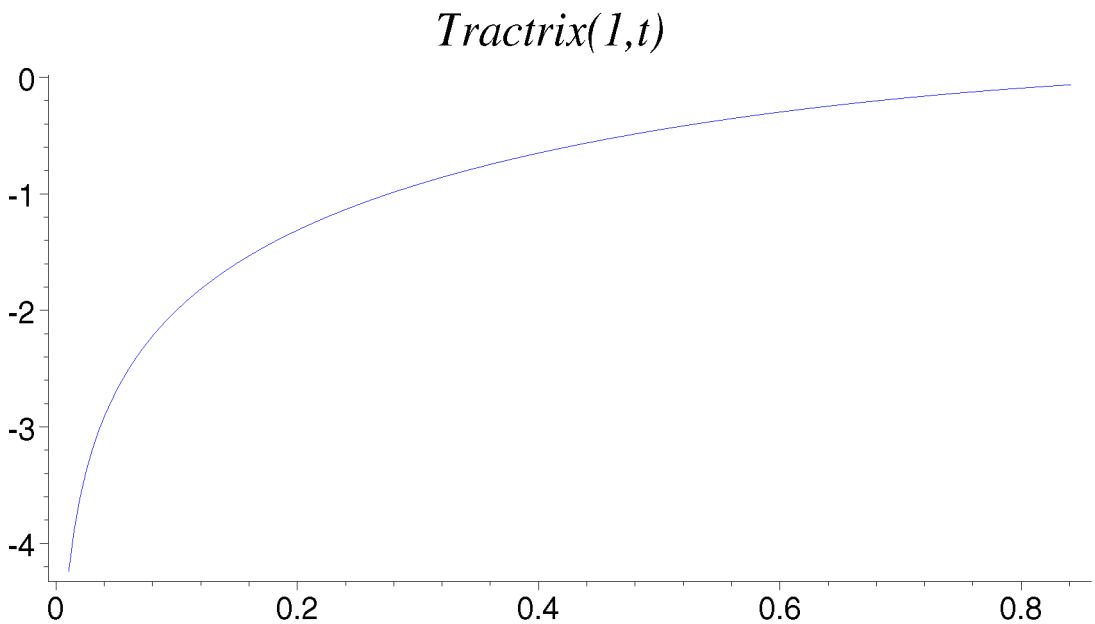


Tractrix (equal length tangent segments)

tractrix = mat(tractrix(a, t))

$$tractrix = \left[\begin{array}{c} a\sin(t) \\ a\left(\cos(t) + \ln\left(\tan\left(\frac{1}{2}t\right)\right)\right) \end{array}\right]$$

curveplot2D(tractrix(1, t), t = 0 .. 1, *title* = "Tractrix(1,t)")



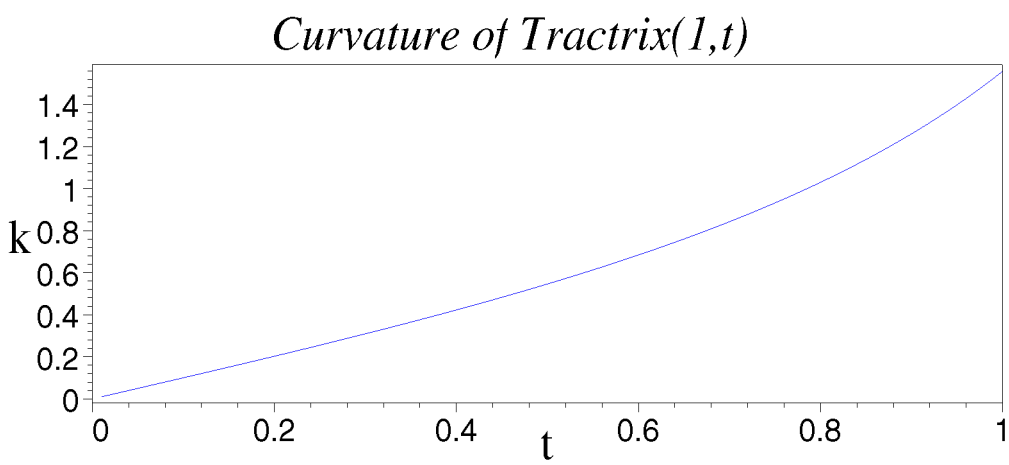
Curvature

κ = simplify(kappa2D(tractrix(a, t), t), *assume* = *real*)

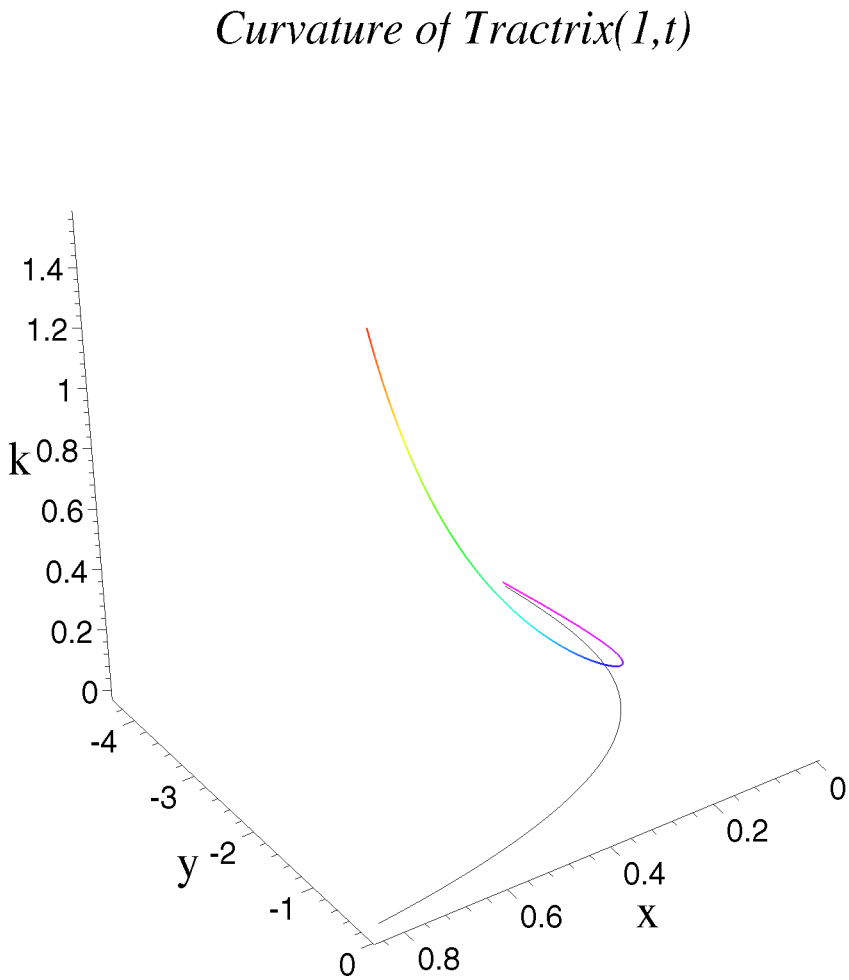
eq := fn(rhs(%), a, t)

$$\kappa = -\frac{\mathrm{signum}(a)\,\mathrm{signum}(\cos(t))\,\mathrm{signum}(\sin(t))\,\sin(t)^3}{a\cos(t)\,(-1+\cos(t)^2)}$$

scalarplot(eq(1, t), t = 0 .. 1, *labels* = ["t", "k"], *title* = "Curvature of Tractrix(1,t)")



curvatureplot3D(tractrix(1, t), eq(1, t), t = 0 .. 1, "Curvature of Tractrix(1,t)", *axes* = *framed*)



Lissajous pattern

$lissajous = \mathrm{mat}(\mathrm{lissajous}(n, m, a, b, \phi, t))$

$$lissajous = \begin{bmatrix} a \sin(n\, t) \\ b \sin(m\, t + \phi) \end{bmatrix}$$

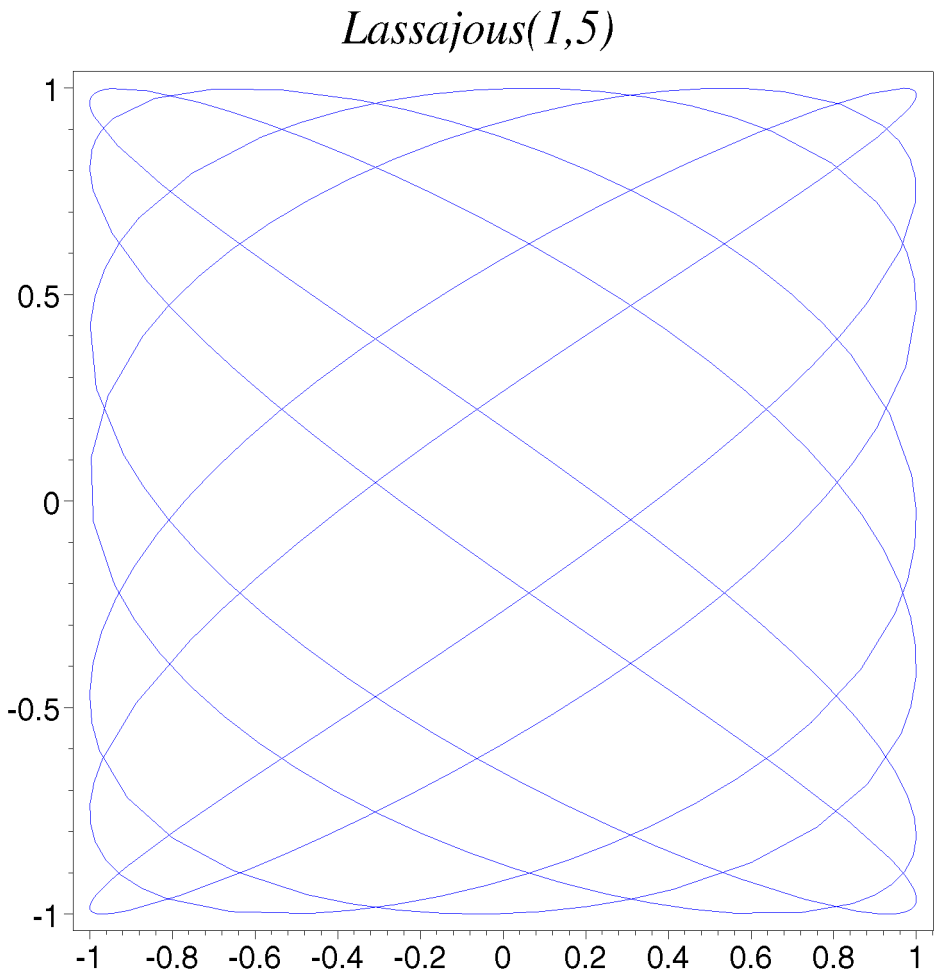
ParametrizedToImplicit(lissajous(n, m, a, b, ϕ, t), ['x', 'y'], t)

isolate(%, y)

applyop(*expand*, [2, 1, 1], %)

$$y = b \sin\left(\frac{m \arcsin\left(\frac{x}{a}\right) + \phi\, n}{n}\right)$$

curveplot2D(lissajous(7, 5, 1, 1, .2 π, t), $t = 0 .. 2\,\pi$, *scaling* = *constrained*, *axes* = *box*, *title* = "Lassajous(1,5)")

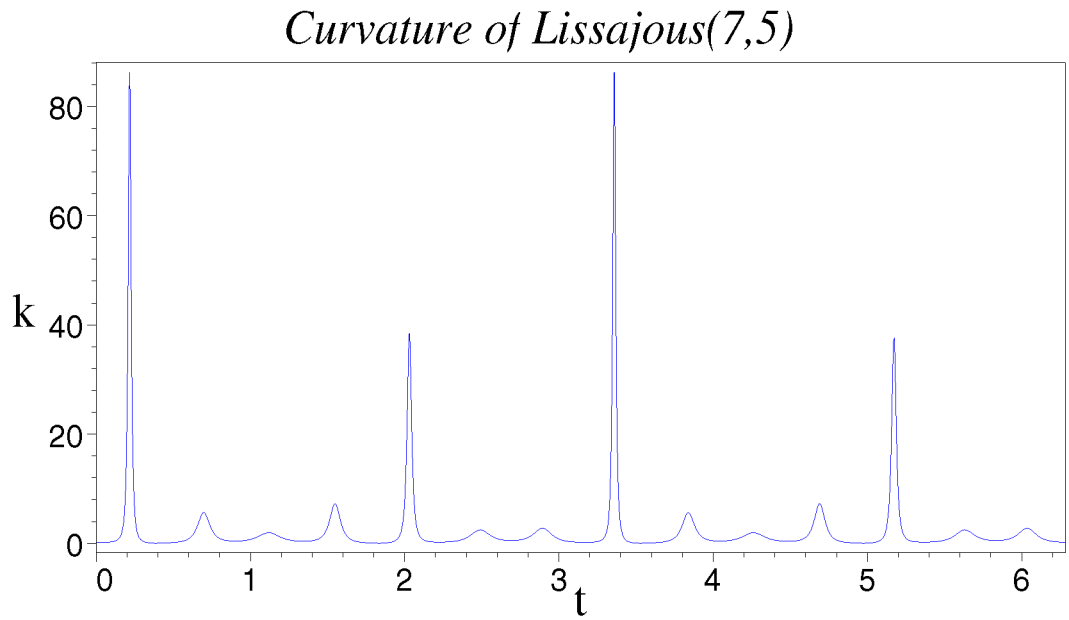


Curvature

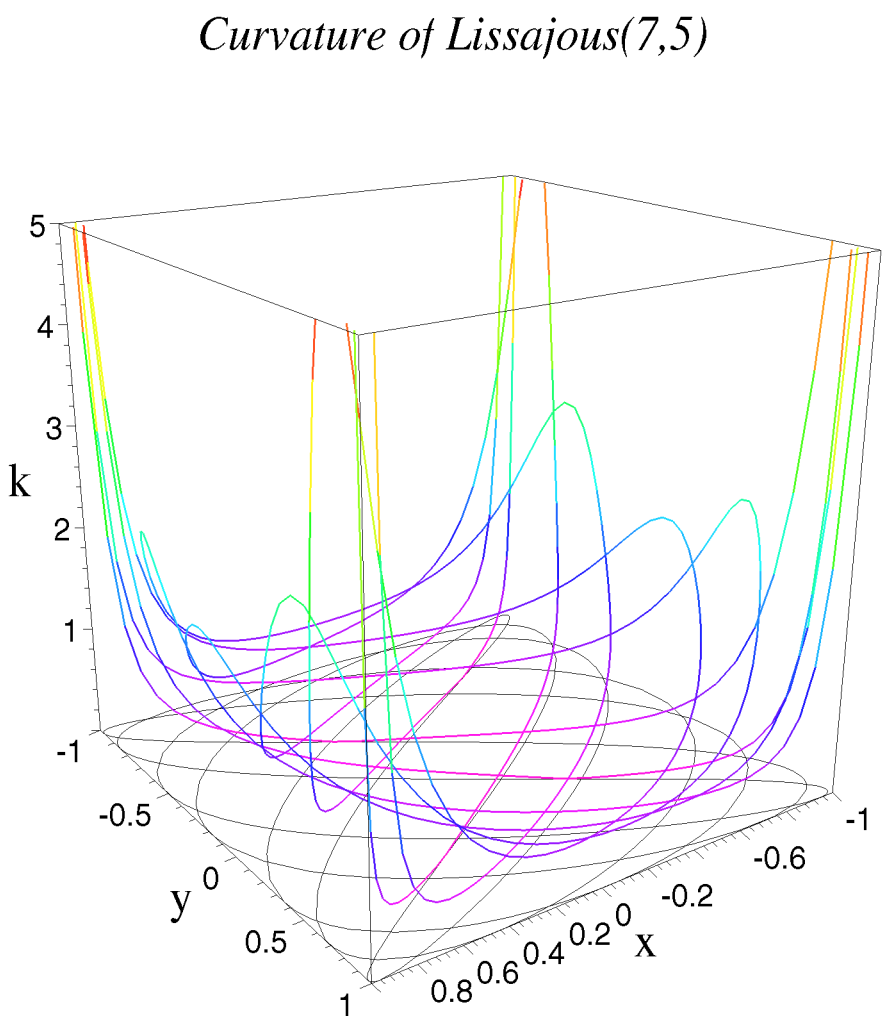
$\kappa = \text{kappa2D}(\text{lissajous}(n, m, a, b, \phi, t), t)$

$$\kappa = \frac{\left| a\, b\, n\, m\, (-\sin(m\, t + \phi)\cos(n\, t)\, m + \cos(m\, t + \phi)\sin(n\, t)\, n) \right|}{(a^2 \cos(n\, t)^2\, n^2 + b^2 \cos(m\, t + \phi)^2\, m^2)^{(3/2)}}$$

scalarplot(kappa2D(lissajous(7, 5, 1, 1, .2 π, t), t), $t = 0 .. 2\,\pi$, *labels* = ["t", "k"], *title* = "Curvature of Lissajous(7,5)", *numpoints* = 500)



curvatureplot3D(lissajous(7, 5, 1, 1, .2 π, *t*), kappa2D(lissajous(7, 5, 1, 1, .2 π, *t*), *t*), *t* = 0 .. 2 π, "Curvature of Lissajous(7,5)", *axes* = *boxed*, *view* = [−1 .. 1, −1 .. 1, 0 .. 5], *numpoints* = 500)

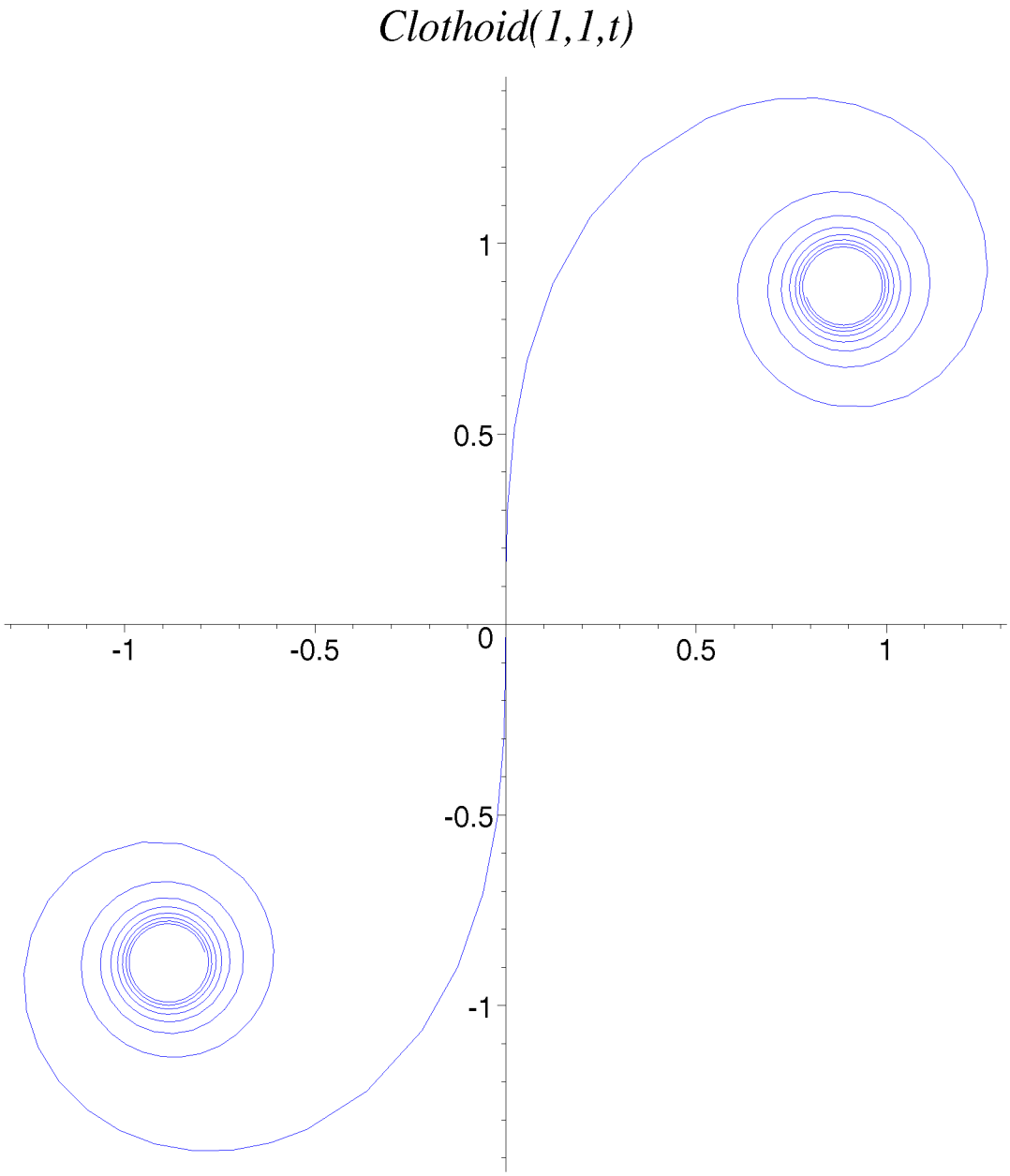


Clothoid (Fresnel integral = Clothoid(1,a))

$clothoid = \mathrm{mat}(\mathrm{clothoid}(n, a, t))$

$$clothoid = \begin{bmatrix} \int_0^t a \sin\left(\frac{u^{(n+1)}}{n+1}\right) du \\ \int_0^t a \cos\left(\frac{u^{(n+1)}}{n+1}\right) du \end{bmatrix}$$

$\mathrm{curveplot2D}(\mathrm{clothoid}(1.0, 1.0, t), t = -10 .. 10, scaling = constrained, axes = normal, title = \text{"Clothoid(1,1,t)"})$



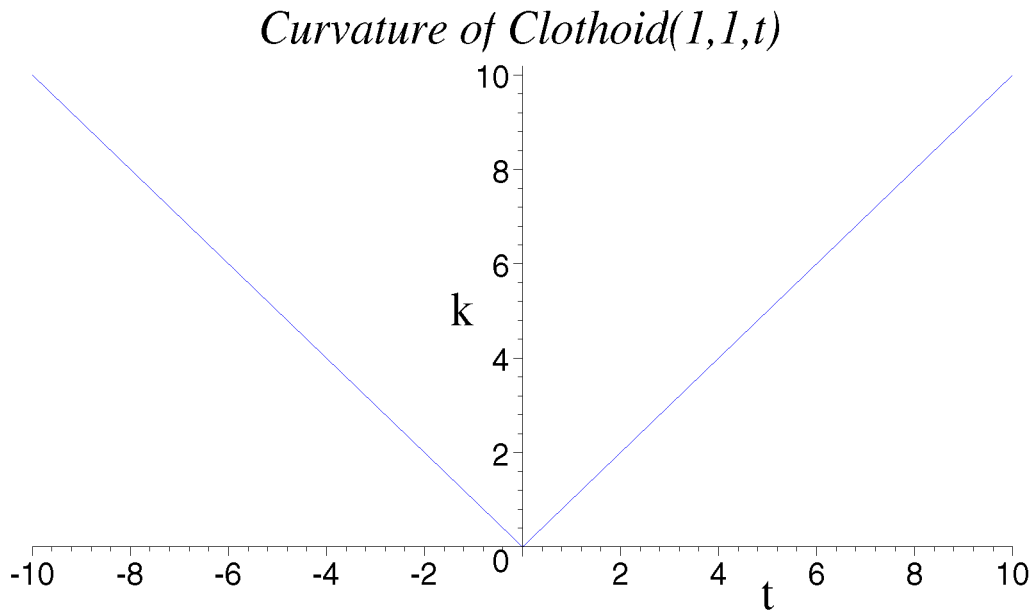
Curvature of a Clothoid

$\kappa = \text{simplify}(\text{kappa2D}(\text{clothoid}(n, a, t), t), \textit{assume} = \textit{real})$

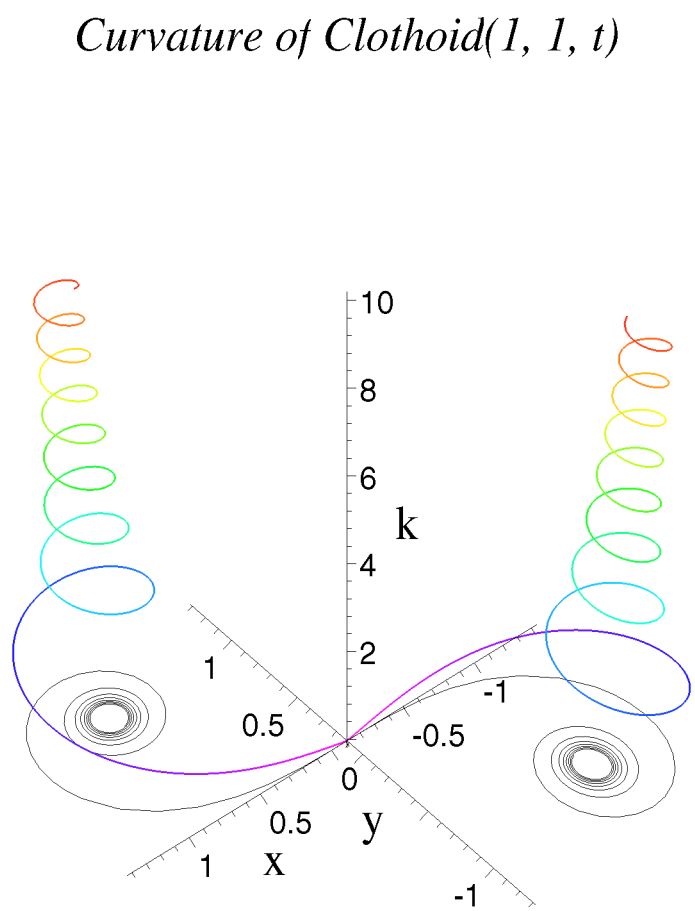
$eq := \text{fn}(\text{rhs}(\%), n, a, t)$

$$\kappa = \frac{|t^n| \, \text{signum}(a)}{a}$$

scalarplot(eq(1, 1, t), $t = -10 .. 10$, *title* = "Curvature of Clothoid(1,1,t)", *labels* = ["t", "k"], *axes* = *normal*)



curvatureplot3D(clothoid(1, 1, *t*), eq(1, 1, *t*), *t* = −10 .. 10, "Curvature of Clothoid(1, 1, t)", *axes* = *normal*, *orientation* = [140, 50], *numpoints* = 500)



Folium of Descartes

$folium = \mathrm{mat}(\mathrm{folium}(t))$

$$folium = \begin{bmatrix} 3\,\dfrac{t}{1+t^3} \\ 3\,\dfrac{t^2}{1+t^3} \end{bmatrix}$$

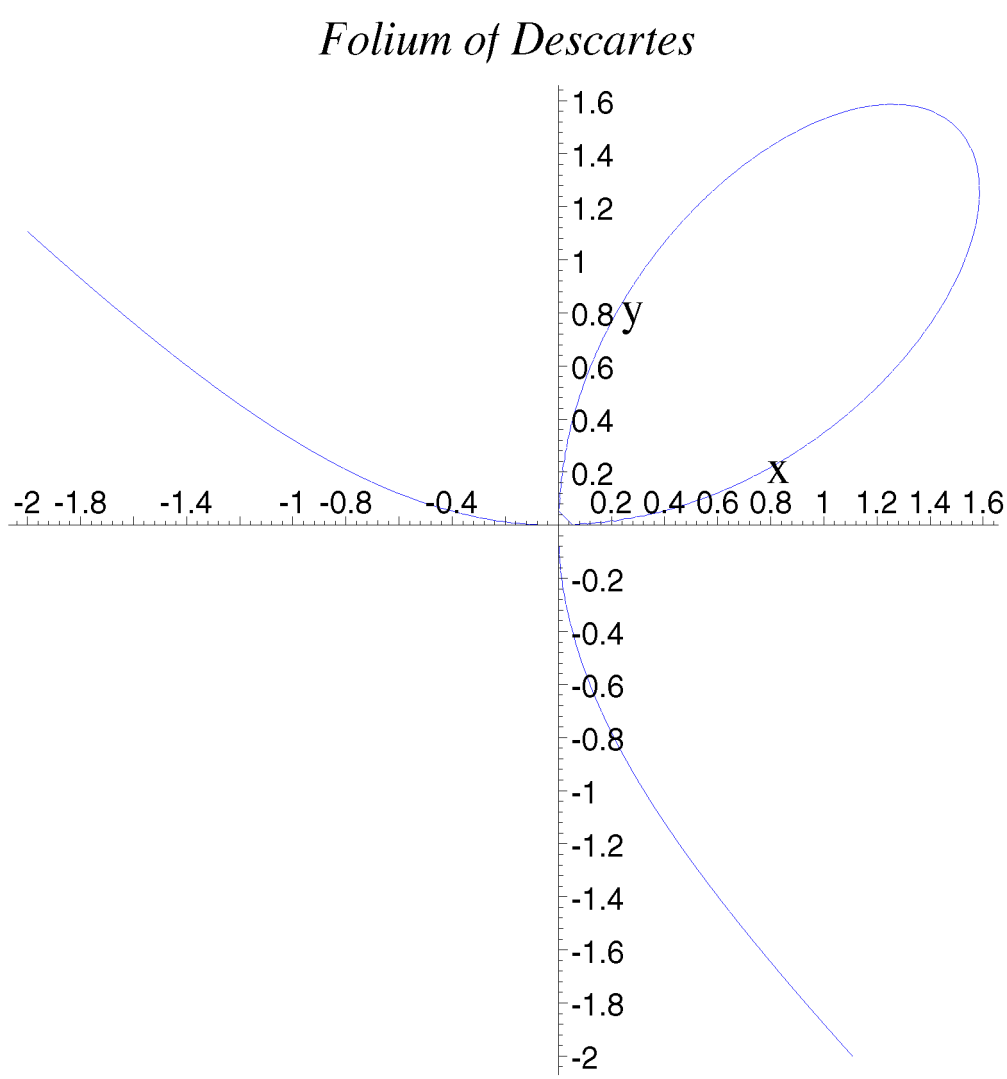
$\mathrm{ParametrizedToImplicit}(\mathrm{folium}(t), [\,'x', 'y'\,], t)$

$FoliumImplicit := \mathrm{fn}(\%, x, y)$

$$x^3 - 3\,y\,x + y^3 = 0$$

The Folium of Descartes is easier to plot in its implicit form:

$\mathrm{implicitplot2D}(\mathrm{FoliumImplicit}(x, y), x = -2 .. 2, y = -2 .. 2, scaling = constrained, axes = normal, grid = [\,75, 75\,], title = \text{"Folium of Descartes"})$



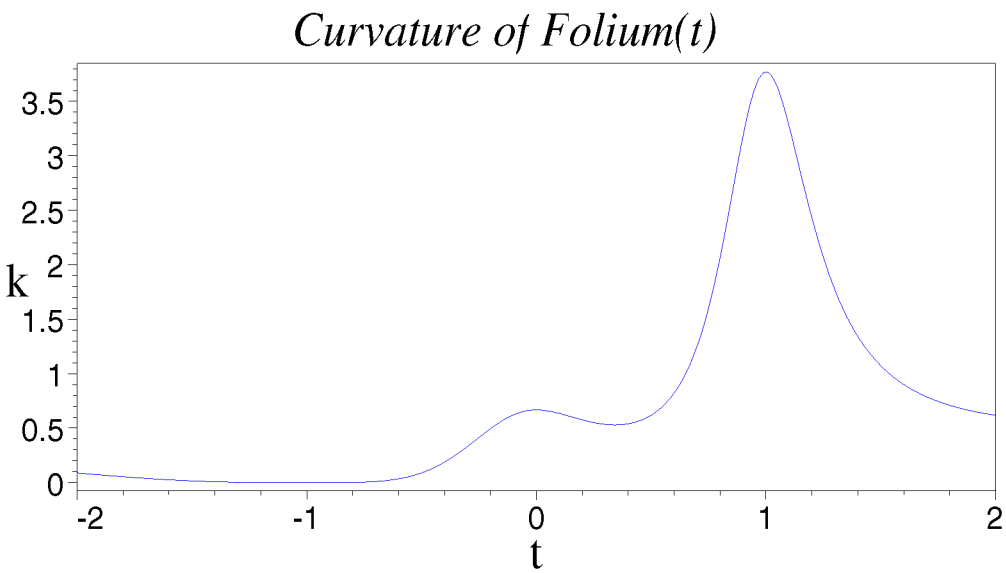
Curvature

$\kappa = \text{simplify}(\text{kappa2D}(\text{folium}(t), t), assume = real)$

$eq := \text{fn}(\text{rhs}(\%), t)$

$$\kappa = \frac{2}{3} \frac{(t+1)^4 (t^2 - t + 1)^4}{(1 - 4t^3 + 4t^6 + 4t^2 - 4t^5 + t^8)^{(3/2)}}$$

$\text{scalarplot}(\text{eq}(t), t = -2 .. 2, labels = [\text{"t"}, \text{"k"}], title = \text{"Curvature of Folium(t)"})$



$F := \text{fn}(\text{folium}(t), t)$

$c1 := \text{curveplot3D}([\text{F}(t)_1, \text{F}(t)_2, \text{eq}(t)], t = -.9 .. 50, numpoints = 1500, thickness = 3)$

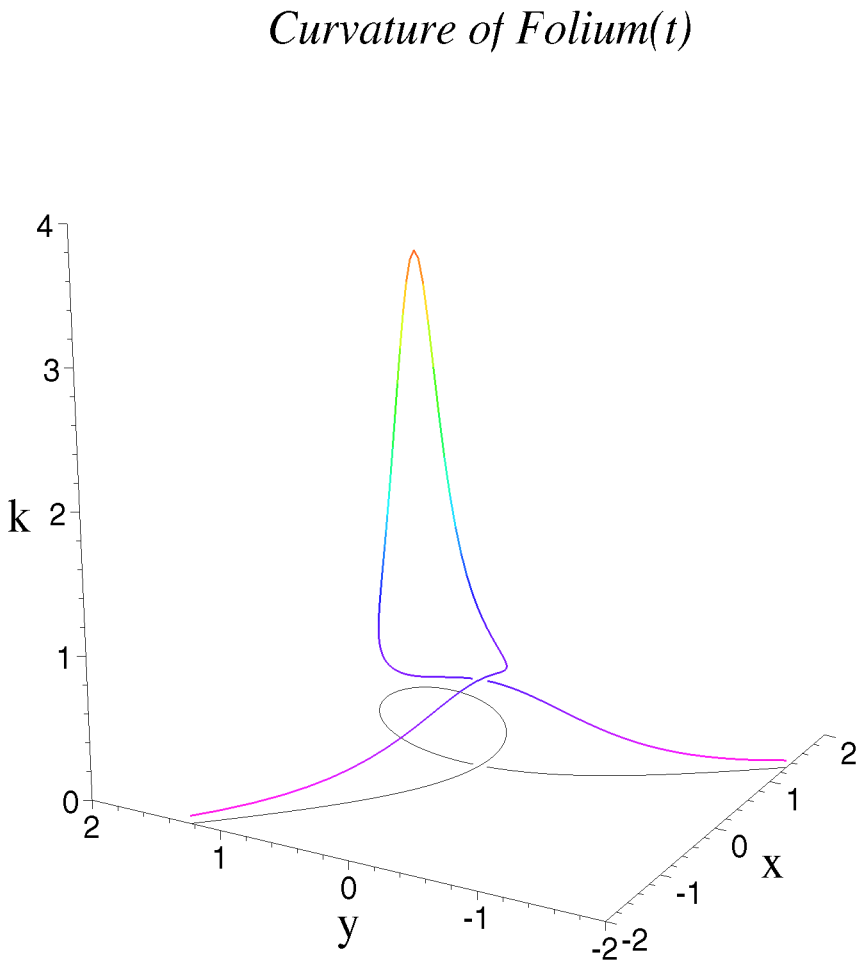
$c2 := \text{curveplot3D}([\text{F}(t)_1, \text{F}(t)_2, \text{eq}(t)], t = -50 .. -1.1, numpoints = 500, thickness = 3)$

$pltx := plottools_{transform}((x, y) \rightarrow [x, y, 0])$

$c3 := \text{curveplot2D}(\text{F}(t), t = -.9 .. 50, numpoints = 1500, color = black)$

$c4 := \text{curveplot2D}(\text{F}(t), t = -50 .. -1.1, numpoints = 500, color = black)$

$plots_{display}(c1, c2, \text{pltx}(c3), \text{pltx}(c4), labels = [\text{"x"}, \text{"y"}, \text{"k"}], title = \text{"Curvature of Folium(t)"},$
$axes = framed, view = [-2 .. 2, -2 .. 2, 0 .. 4], projection = .8, orientation = [-150, 80])$



Cassinian Oval

cassinian(a, b, x, y)

$$(x^2+y^2+a^2)^2-4\,a^2\,x^2-b^4$$

$tmp := (x^2+y^2+a^2)^2-4\,a^2\,x^2-b^4$

isolate(tmp, y^2)

$$y^2=\sqrt{4\,a^2\,x^2+b^4}-x^2-a^2$$

?

?

?

?

cassinian(a, b, t)

$$[(a^2\cos(2\,t)+\sqrt{b^4-a^4\sin(2\,t)^2})\cos(t),\,(a^2\cos(2\,t)+\sqrt{b^4-a^4\sin(2\,t)^2})\sin(t)]$$

$$\left((((a^2\cos(2\,t)+\sqrt{b^4-a^4\sin(2\,t)^2})\cos(t))^2+((a^2\cos(2\,t)+\sqrt{b^4-a^4\sin(2\,t)^2})\sin(t))^2+a^2\right)^2$$
$$-4\,a^2\,((a^2\cos(2\,t)+\sqrt{b^4-a^4\sin(2\,t)^2})\cos(t))^2-b^4$$

Collect(simplify(%), [$\sqrt{b^4-a^4+a^4\cos(2\,t)^2}$, cos($2\,t$)], *loc*, *factor*)

$$(8\,a^6\cos(2\,t)^3-4\,a^2\,(-a^2-b^4+a^4+2\cos(t)^2\,a^2)\cos(2\,t))\sqrt{b^4-a^4+a^4\cos(2\,t)^2}$$
$$+8\,a^8\cos(2\,t)^4-4\,a^4\,(2\cos(t)^2\,a^2+2\,a^4-2\,b^4-a^2)\cos(2\,t)^2$$
$$-(b-a)\,(b+a)\,(b^2+a^2)\,(-b^4+1+4\cos(t)^2\,a^2-2\,a^2+a^4)$$

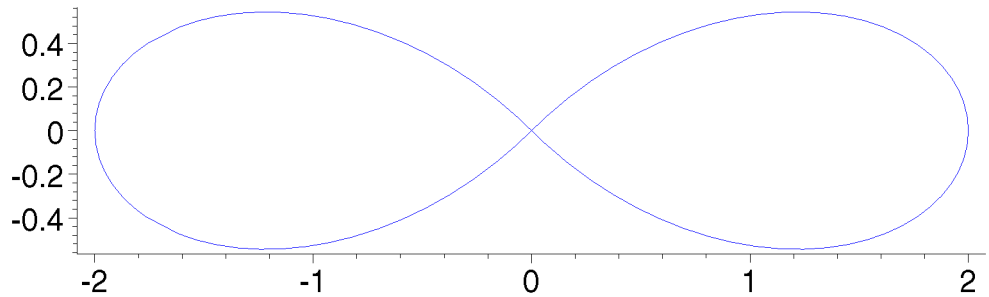
Sqrt[M/2] {Cos[t], Sin[t]}, where M is 2 a^2 Cos[2 t] + 2 Sqrt[(-a^4+b^4) + a^4 Cos[2 t]^2], 0 < t <= 2 Pi, and a < b.

build_curves( )

```
Warning, new definition for cassinian
```

[*J*, *ParametrizedToImplicit*, *astroid*, *binormal*, *cardioid*, *cassinian*, *catenary*, *circle*, *cissoid*, *clotharg*, *clothoid*, *cycloid*, *deltoid*, *ellipse*, *folium*, *helix*, *hyperbola*, κ, *kappa2D*, *lemniscate*, *lissajous*, *logspiral*, *parabola*, *tangent*, *torsion*, *torusknot*, *tractrix*, *twicubic*, *unitnormal*]

curveplot2D(cassinian(1, 1, t), $t = 0$ .. $2\,\pi$, *scaling* = *constrained*)



factormat(cassinian(a, b, t), *factor*)

$$(a^2 \cos(2\,t) + \sqrt{-(a^2 \sin(2\,t) - b^2)\,(a^2 \sin(2\,t) + b^2)}\,)\,[\cos(t), \sin(t)]$$

cassinian(a, b, t)

$$[(a^2 \cos(2\,t) + \sqrt{b^4 - a^4 \sin(2\,t)^2}\,)\cos(t), (a^2 \cos(2\,t) + \sqrt{b^4 - a^4 \sin(2\,t)^2}\,)\sin(t)]$$

ParametrizedToImplicit(cassinian(a, b, t), [x, y], t)

```
Warning, computation interrupted
```

subs $\left(x = \text{cassinian}(a, b, t)_1, y = \text{cassinian}(a, b, t)_2, (x^2 + y^2 + a^2)^2 - 4\,a^2\,x^2 - b^4\right)$

$$\left((a^2 \cos(2\,t) + \sqrt{b^4 - a^4 \sin(2\,t)^2}\,)^2 \cos(t)^2 + (a^2 \cos(2\,t) + \sqrt{b^4 - a^4 \sin(2\,t)^2}\,)^2 \sin(t)^2 + a^2\right)^2 - 4\,(a^2 \cos(2\,t) + \sqrt{b^4 - a^4 \sin(2\,t)^2}\,)^2 \cos(t)^2\,a^2 - b^4$$

simplify(%, *assume* = *real*)

$$-8\,a^8 \cos(2\,t)^2 + 2\,a^2\,b^4 + 8\,b^4\,a^4 \cos(2\,t)^2 + a^4 - b^4 + b^8 - 2\,b^4\,a^4 + a^8 - 2\,a^6 - 8\cos(t)^2\,a^4 \cos(2\,t)\sqrt{b^4 - a^4 + a^4 \cos(2\,t)^2} - 8\cos(t)^2\,a^6 \cos(2\,t)^2 - 4\cos(t)^2\,a^2\,b^4 + 4\cos(t)^2\,a^6 + 8\,a^6 \cos(2\,t)^3 \sqrt{b^4 - a^4 + a^4 \cos(2\,t)^2} + 4\,a^4 \cos(2\,t)\sqrt{b^4 - a^4 + a^4 \cos(2\,t)^2}$$

$$+4\,a^2\cos(2\,t)\sqrt{b^4-a^4+a^4\cos(2\,t)^2}\;b^4-4\,a^6\cos(2\,t)\sqrt{b^4-a^4+a^4\cos(2\,t)^2}$$
$$+8\,a^8\cos(2\,t)^4+4\,a^6\cos(2\,t)^2$$

Collect(%, [sqrt, $b^4-a^4+a^4\cos(2\,t)^2$], *factor*, *loc*)

$$4\,a^2\cos(2\,t)\,(-2\cos(t)^2\,a^2+2\,a^4\cos(2\,t)^2+a^2+b^4-a^4)\sqrt{b^4-a^4+a^4\cos(2\,t)^2}$$
$$-8\,a^8\cos(2\,t)^2+2\,a^2\,b^4+8\,b^4\,a^4\cos(2\,t)^2+a^4-b^4+b^8-2\,b^4\,a^4+a^8-2\,a^6+8\,a^8\cos(2\,t)^4$$
$$-8\cos(t)^2\,a^6\cos(2\,t)^2-4\cos(t)^2\,a^2\,b^4+4\cos(t)^2\,a^6+4\,a^6\cos(2\,t)^2$$

?

?

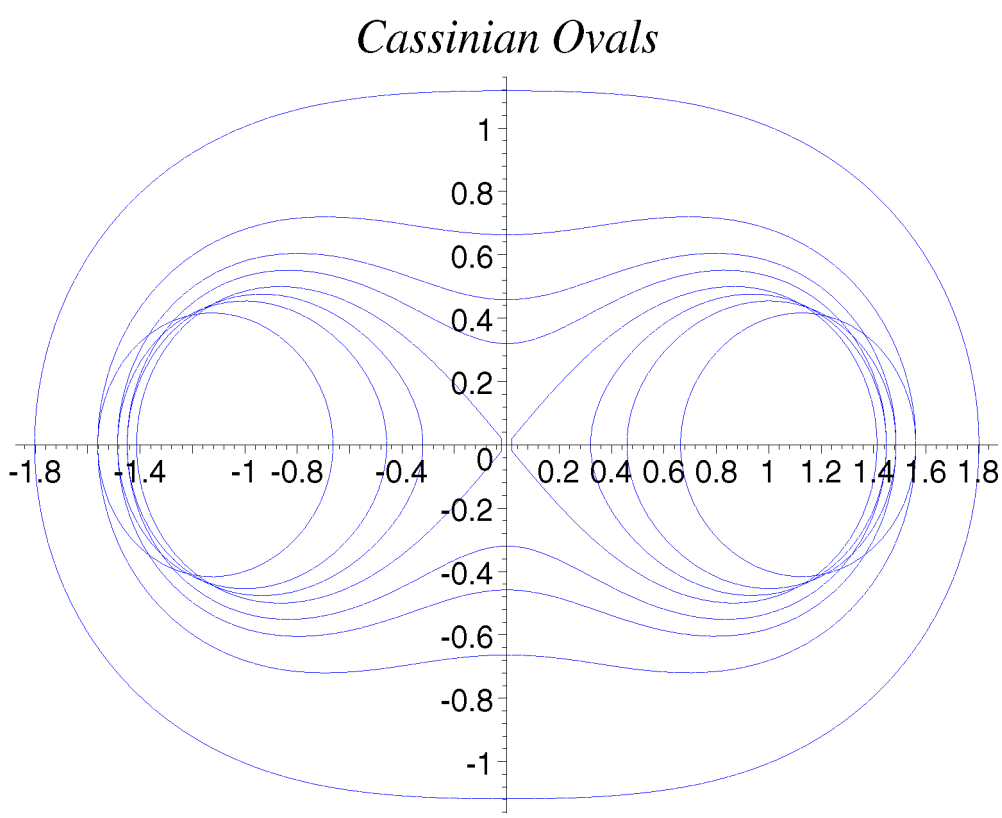
?

?

CassiniPlot := $(p, q) \rightarrow$ implicitplot2D(subs($a = p$, $b = q$, cassinian(a, b, x, y)), $x = -2$ .. 2, $y = -2$ .. 2, *scaling* = *constrained*, *axes* = *normal*, *grid* = [100, 100])

plts := [CassiniPlot(1.0, 1.0), CassiniPlot(1, 1.05), CassiniPlot(1, 1.1), CassiniPlot(1, 1.2), CassiniPlot(1, 1.5), CassiniPlot(1, 2), CassiniPlot(1, 2.5), CassiniPlot(1, 3)]

$plots_{display}$(*plts*, *axes* = *normal*, *title* = "Cassinian Ovals")

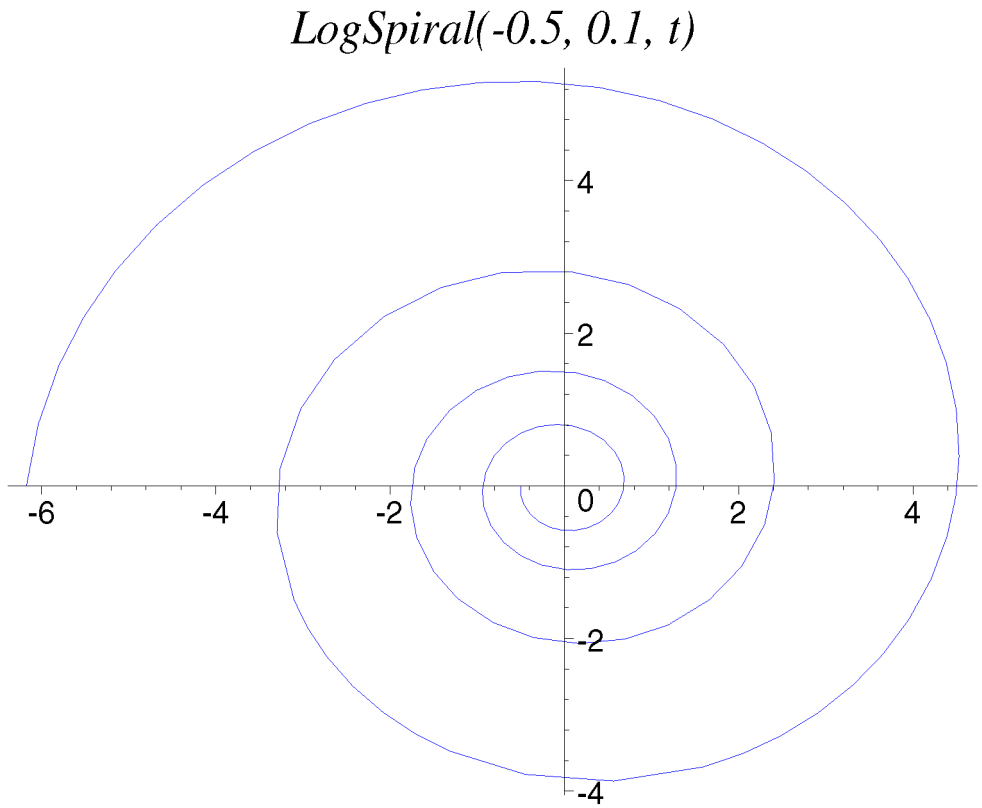


Log Spiral

$logspiral = \mathrm{mat}(\mathrm{logspiral}(a, b, t))$

$$logspiral = \begin{bmatrix} a\,\mathbf{e}^{(b\,t)}\cos(t) \\ a\,\mathbf{e}^{(b\,t)}\sin(t) \end{bmatrix}$$

$\mathrm{curveplot2D}(\mathrm{logspiral}(-.5, .1, t), t = 0 .. 8\,\pi, axes = normal, title = \text{"LogSpiral(-0.5, 0.1, t)"})$

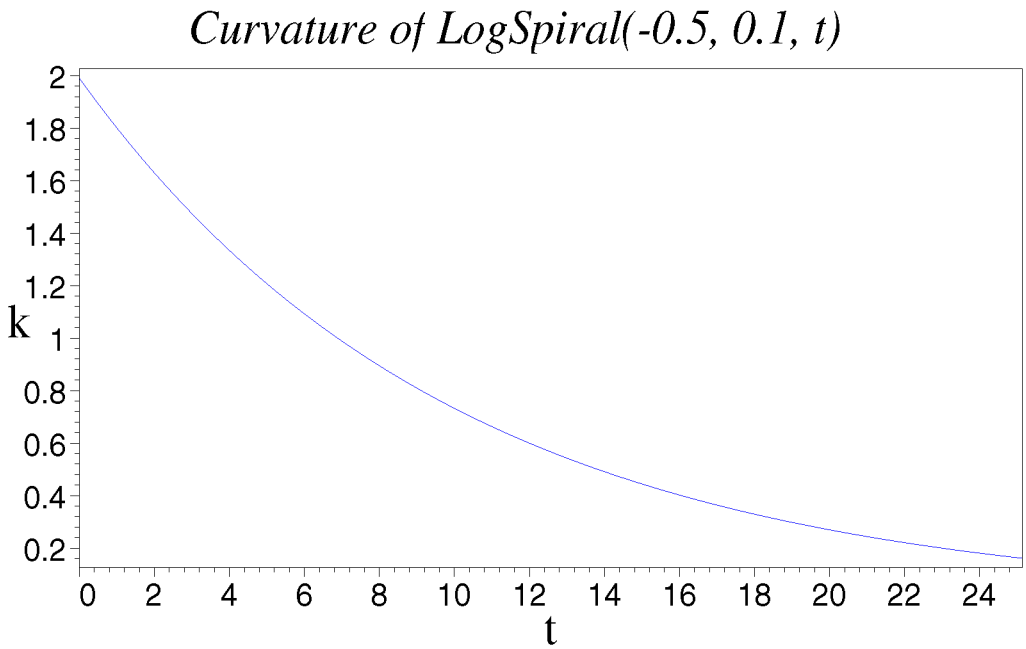


Curvature

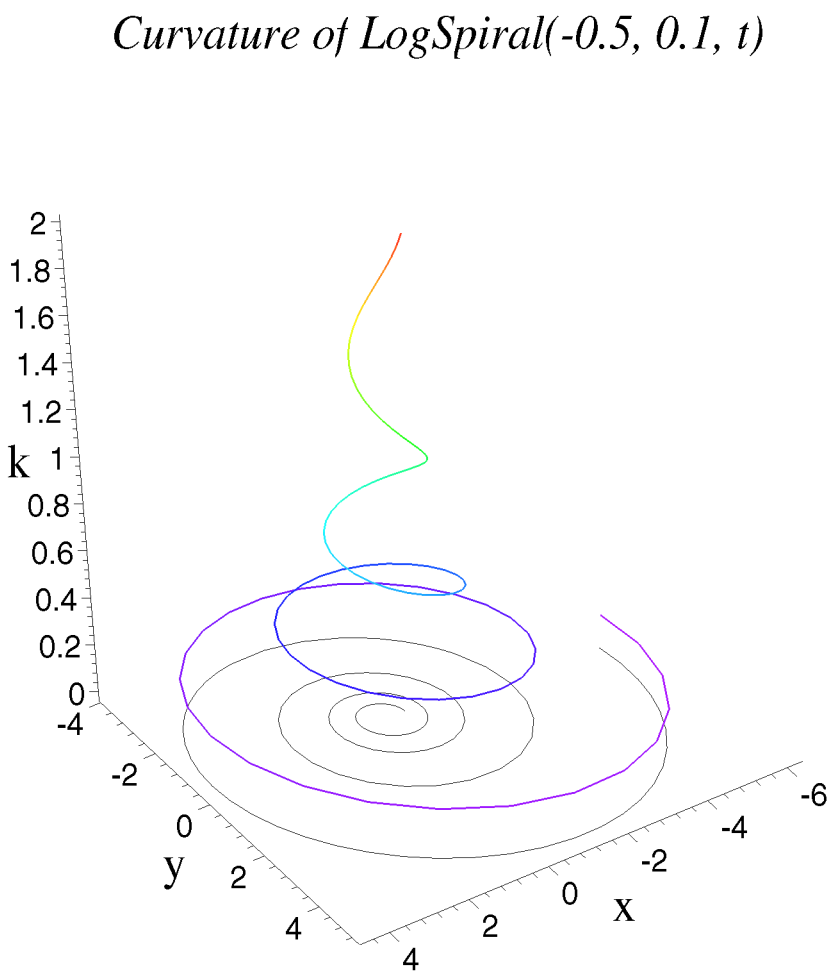
$\kappa = \mathrm{simplify}(\mathrm{kappa2D}(\mathrm{logspiral}(a, b, t), t), assume = real)$

$$\kappa = \frac{\mathrm{signum}(a)\,\mathbf{e}^{(-b\,t)}}{a\sqrt{b^2+1}}$$

$\mathrm{scalarplot}(\mathrm{kappa2D}(\mathrm{logspiral}(-.5, .1, t), t), t = 0 .. 8\,\pi, title = \text{"Curvature of LogSpiral(-0.5, 0.1, t)"}, labels = [\text{"t"}, \text{"k"}])$



curvatureplot3D(logspiral(−.5, .1, t), kappa2D(logspiral(−.5, .1, t), t), $t = 0$ .. 8 π, "Curvature of LogSpiral(-0.5, 0.1, t)", *axes* = *framed*)



Catenary

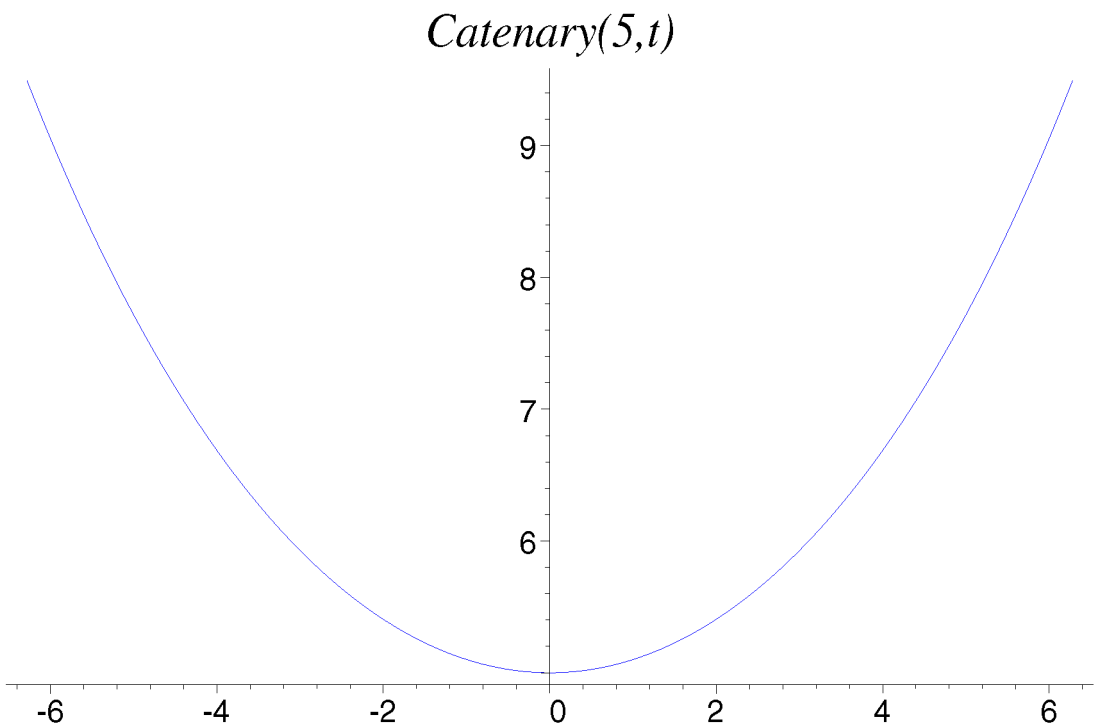
$catenary = \mathrm{mat}(\mathrm{catenary}(a, t))$

$$catenary = \begin{bmatrix} t \\ a \cosh\left(\frac{t}{a}\right) \end{bmatrix}$$

$\mathrm{isolate}(\mathrm{ParametrizedToImplicit}(\mathrm{catenary}(a, t), [\text{'}x\text{'}, \text{'}y\text{'}], t), y)$

$$y = a \cosh\left(\frac{x}{a}\right)$$

$\mathrm{curveplot2D}(\mathrm{catenary}(5, t), t = -2\,\pi .. 2\,\pi, axes = normal, title = \text{"Catenary(5,t)"})$

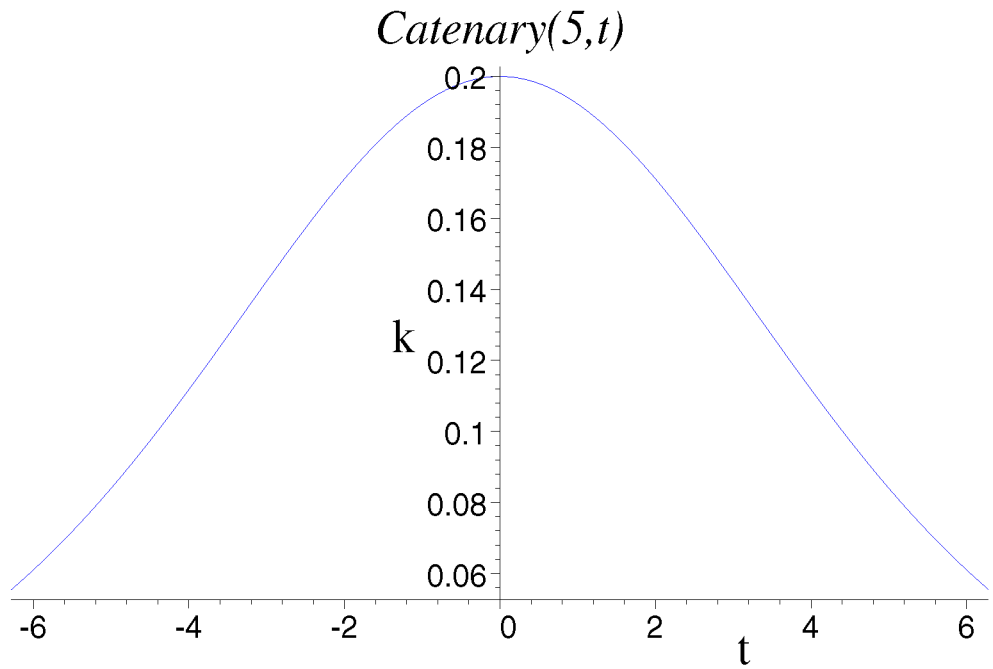


Curvature

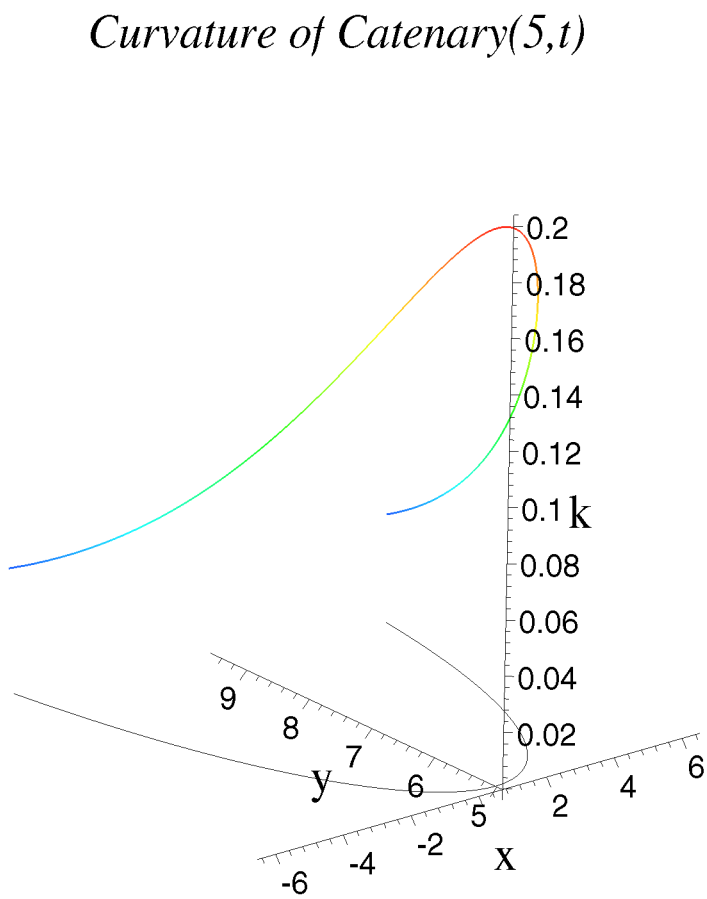
$\kappa = \mathrm{simplify}(\mathrm{kappa2D}(\mathrm{catenary}(a, t), t), assume = real)$

$$\kappa = \frac{1}{\cosh\left(\frac{t}{a}\right)^2 |a|}$$

$\mathrm{scalarplot}(\mathrm{kappa2D}(\mathrm{catenary}(5, t), t), t = -2\,\pi .. 2\,\pi, axes = normal, labels = [\text{"t"}, \text{"k"}], title = \text{"Catenary(5,t)"})$



curvatureplot3D(catenary(5, t), kappa2D(catenary(5, t), t), $t = -2\,\pi$ .. $2\,\pi$, "Curvature of Catenary(5,t)", *axes* = *normal*, *orientation* = [−125, 80])



Astroid

$astroid = \mathrm{mat}(\mathrm{astroid}(n, a, b, t))$

$$astroid = \begin{bmatrix} (a\cos(t))^n \\ (b\sin(t))^n \end{bmatrix}$$

$\mathrm{eliminate}(\{x = \mathrm{astroid}(n, a, b, t)_1, y = \mathrm{astroid}(n, a, b, t)_2\}, t)$

$$\mathrm{select}\left(has, \{\%\}, \arctan\left(\frac{\mathbf{e}^{\left(\frac{\ln(y)}{n}\right)}}{b}, \frac{\mathbf{e}^{\left(\frac{\ln(x)}{n}\right)}}{a}\right)\right)$$

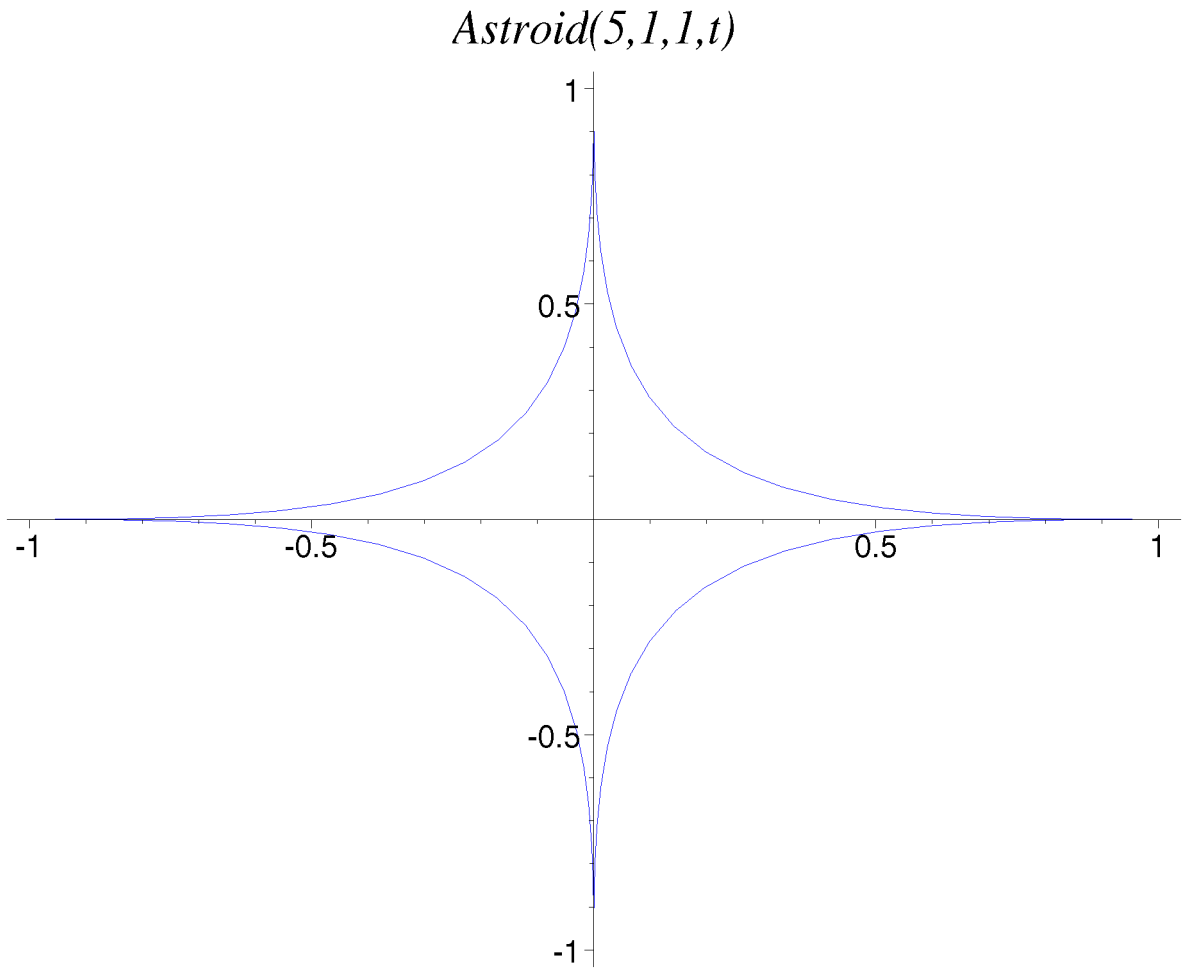
$\mathrm{op}([1, 2, 1], \%) = 0$

$$\mathrm{collect}\left(\frac{\%}{(a\,b)^2}, [a, b], simplify\right)$$

$\mathrm{map}(x \rightarrow x + 1, \%)$

$$\frac{y^{\left(2\frac{1}{n}\right)}}{b^2} + \frac{x^{\left(2\frac{1}{n}\right)}}{a^2} = 1$$

$\mathrm{curveplot2D}(\mathrm{astroid}(5, 1, 1, t), t = 0 .. 2\pi, axes = normal, title = \text{"Astroid(5,1,1,t)"})$

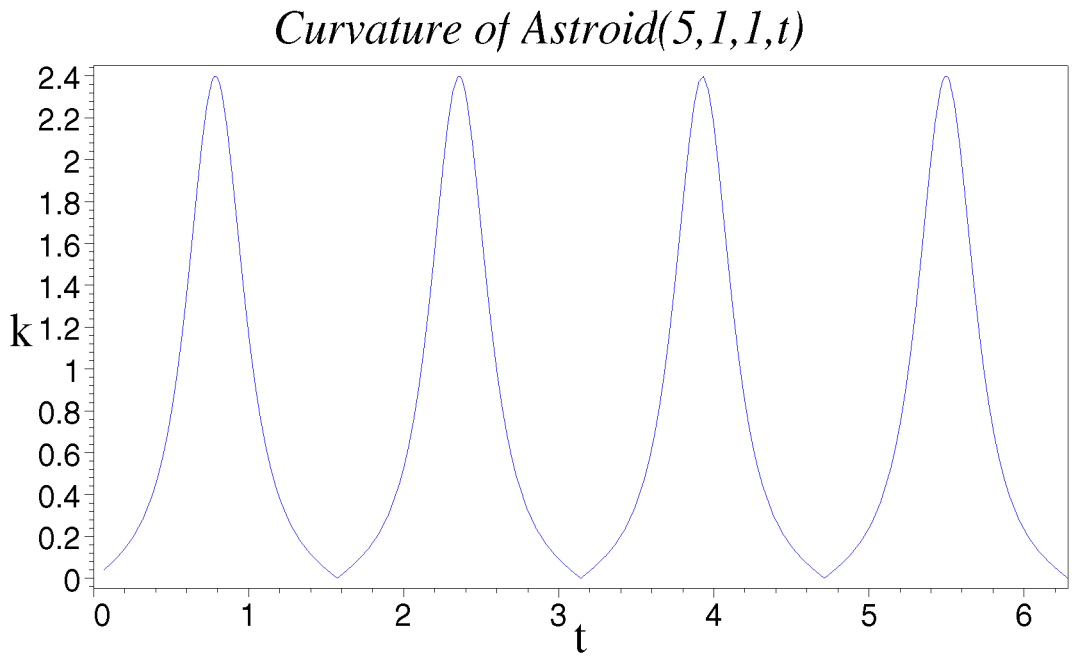


Curvature

$\kappa = \text{kappa2D}(\text{astroid}(n, a, b, t), t)$

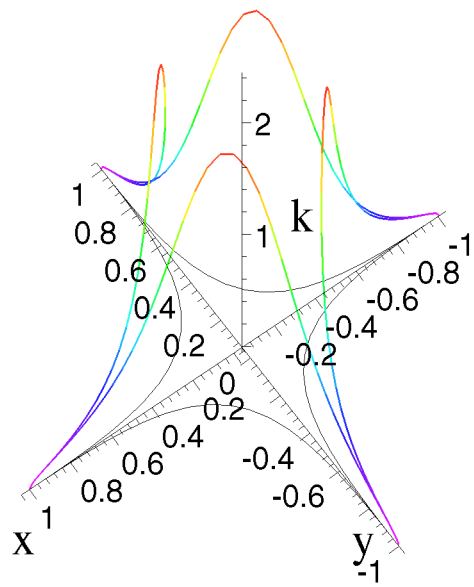
$$\kappa = \frac{\left| \frac{(a\cos(t))^n n^2 (b\sin(t))^n (-2+n)}{\cos(t)\sin(t)} \right|}{\left(\frac{n^2 \left(((a\cos(t))^n)^2 \sin(t)^4 + ((b\sin(t))^n)^2 \cos(t)^4 \right)}{\cos(t)^2 \sin(t)^2} \right)^{(3/2)}}$$

scalarplot(kappa2D(astroid(5, 1, 1, t), t), $t = 0 .. 2\pi$, *title* = "Curvature of Astroid(5,1,1,t)", *labels* = ["t", "k"])



curvatureplot3D(astroid(5, 1, 1, t), kappa2D(astroid(5, 1, 1, t), t), $t = 0 .. 2\pi$, "Curvature of Astroid(5,1,1,t)", *axes* = *normal*, *orientation* = [144, 27], *numpoints* = 200)

Curvature of Astroid(5,1,1,t)



Cardioid

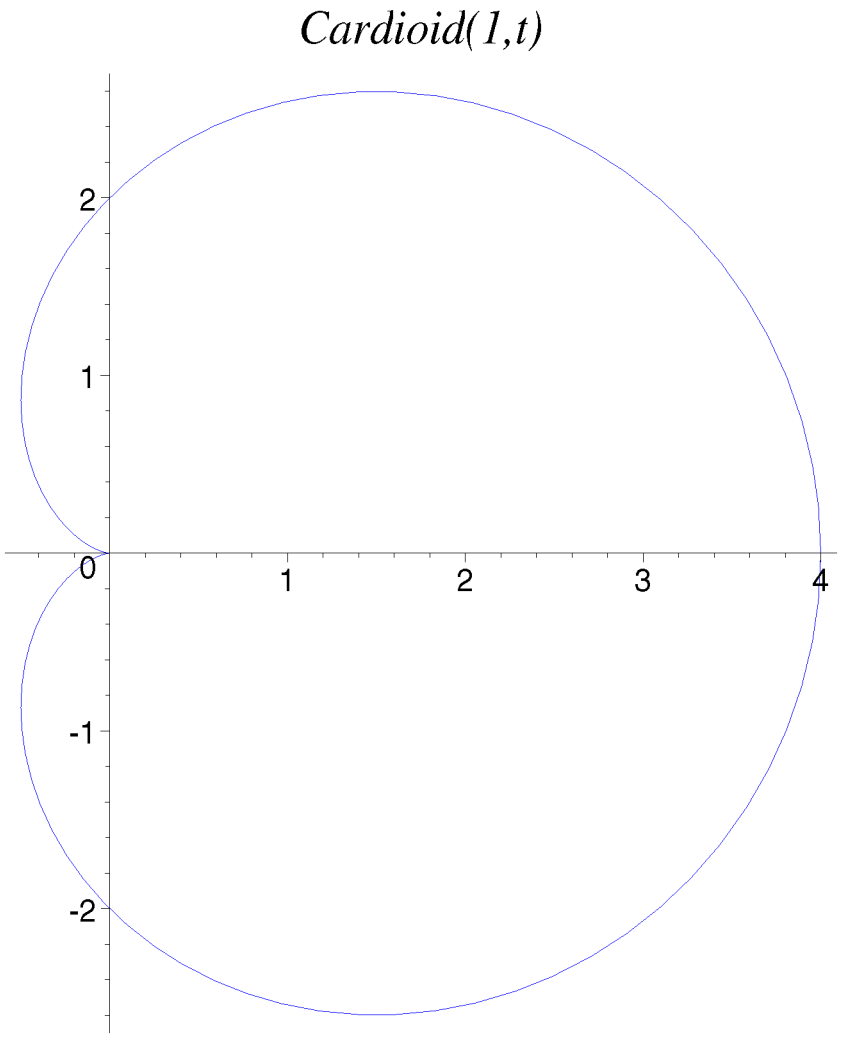
$cardioid = \mathrm{mat}(\mathrm{cardioid}(a, t))$

$$cardioid = \begin{bmatrix} 2\,a\cos(t)\,(\cos(t)+1) \\ 2\,a\sin(t)\,(\cos(t)+1) \end{bmatrix}$$

$\mathrm{ParametrizedToImplicit}(\mathrm{cardioid}(a, t), [\,'x', 'y'\,], t)$

$$(4\,a\,y^2\,x - 4\,x^3\,a + x^4 + 2\,y^2\,x^2 + y^4)\,(-4\,y^2\,a^2 - 4\,x^3\,a + x^4 - 4\,a\,y^2\,x + 2\,y^2\,x^2 + y^4) = 0$$

$\mathrm{curveplot2D}(\mathrm{cardioid}(1, t), t = 0 .. 2\,\pi, axes = normal, scaling = constrained, title = \text{"Cardioid(1,t)"})$

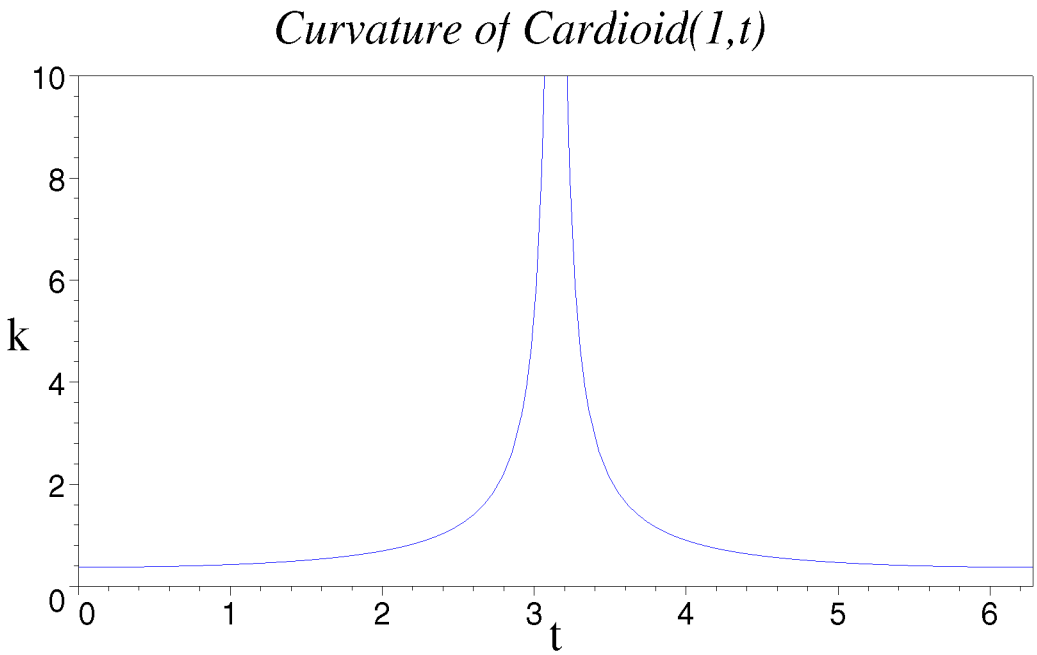


Curvature

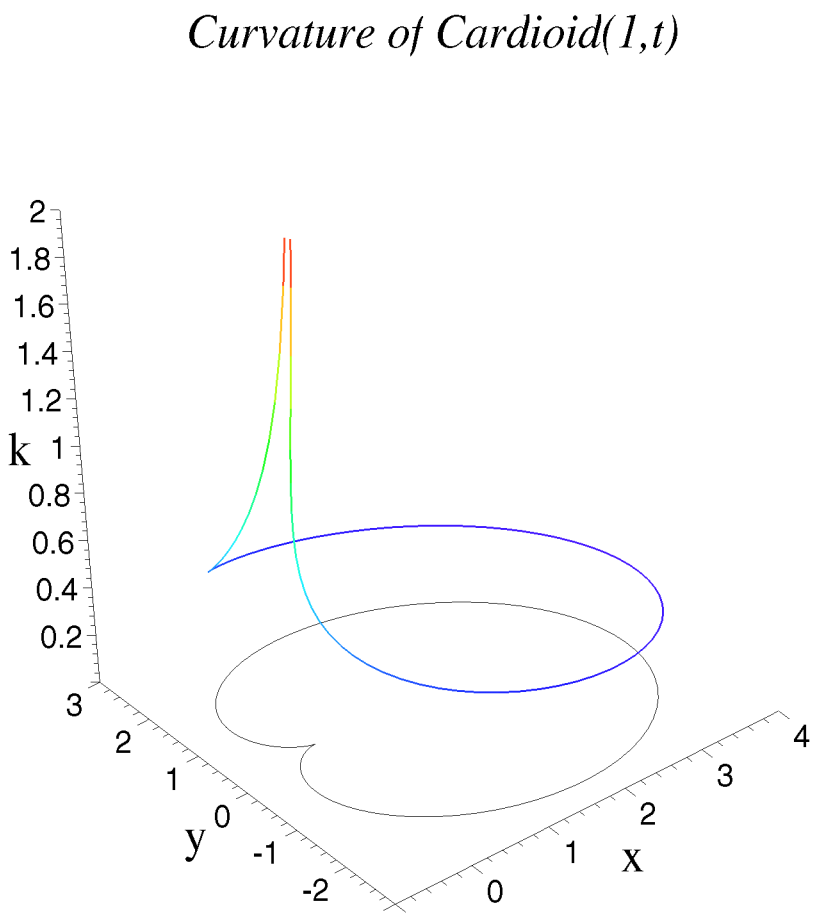
$\kappa = \text{simplify}(\text{kappa2D}(\text{cardioid}(a, t), t), \textit{assume} = \textit{real})$

$$\kappa = \frac{3}{8} \frac{\sqrt{2}\ \text{signum}(a)}{a\sqrt{1+\cos(t)}}$$

$\text{scalarplot}(\text{kappa2D}(\text{cardioid}(1, t), t), t = 0 .. 2\pi, \textit{view} = 0 .. 10, \textit{title} = \text{"Curvature of Cardioid(1,t)"}, \textit{labels} = [\text{"t"}, \text{"k"}])$



curvatureplot3D(cardioid(1, t), kappa2D(cardioid(1, t), t), $t = 0 .. 2\pi$, "Curvature of Cardioid(1,t)", *axes* = *framed*, *orientation* = [−130, 70], *view* = [−1 .. 4, −3 .. 3, 0 .. 2])

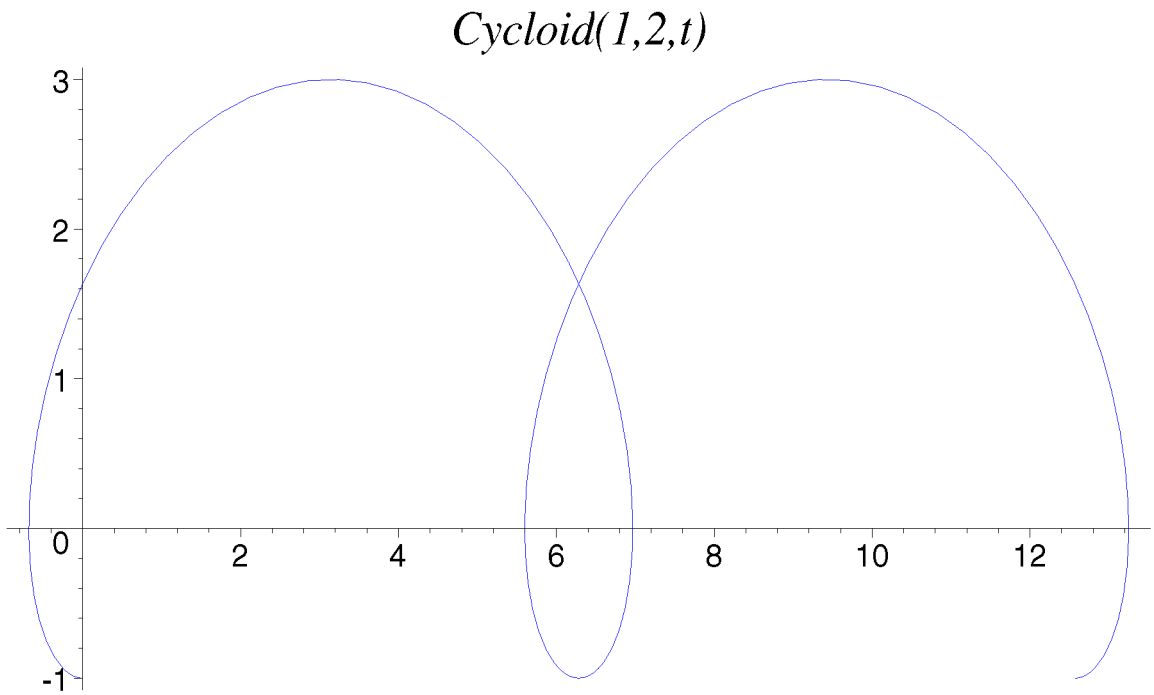


Cycloid

$cycloid = \mathrm{mat}(\mathrm{cycloid}(a, b, t))$

$$cycloid = \begin{bmatrix} a\,t - b\sin(t) \\ a - b\cos(t) \end{bmatrix}$$

curveplot2D(cycloid(1, 2, t), $t = 0 .. 4\,\pi$, *axes* = *normal*, *title* = "Cycloid(1,2,t)")

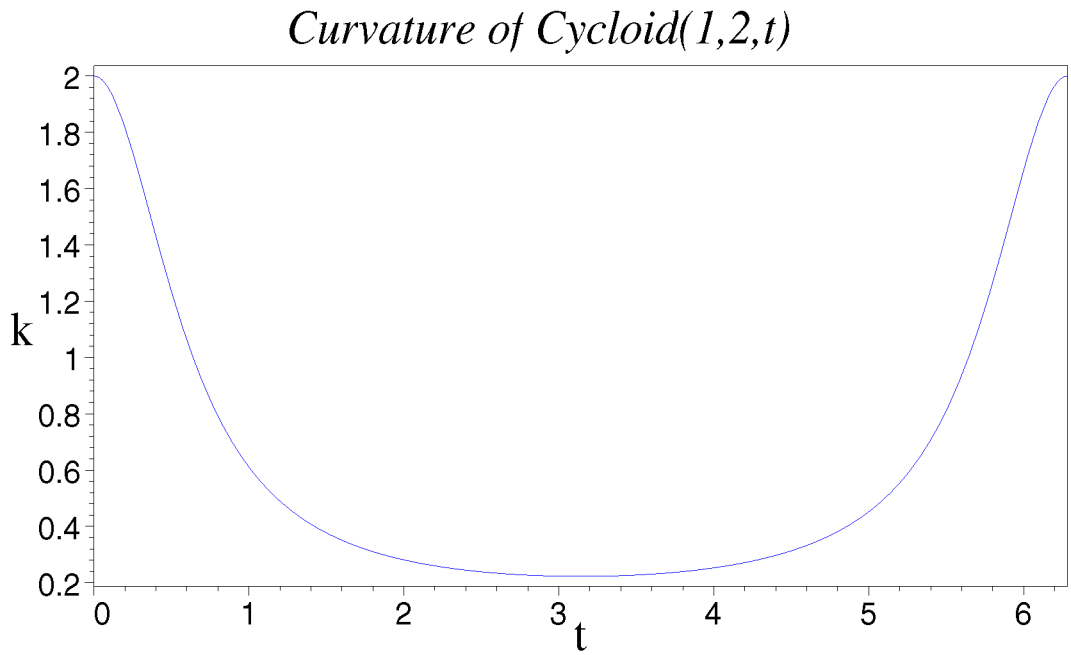


Curvature

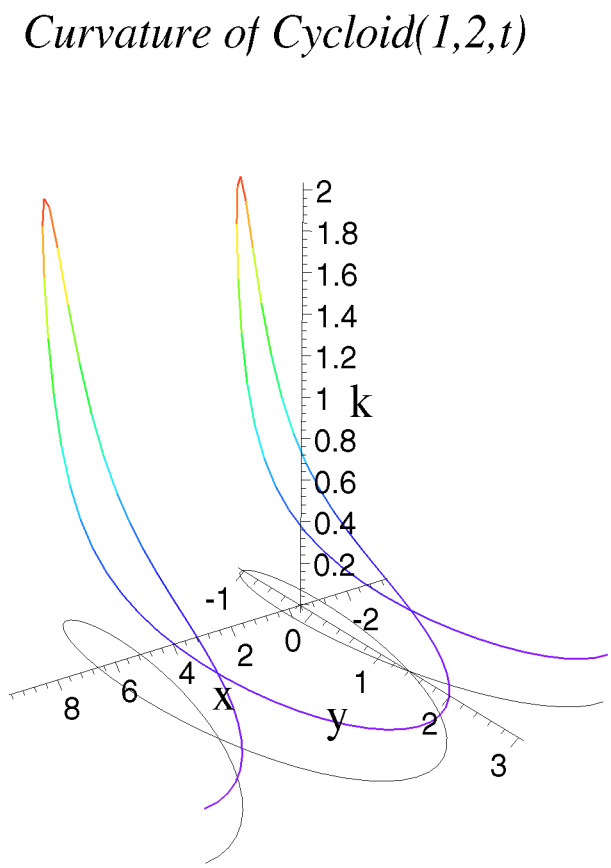
$\kappa = \mathrm{simplify}(\mathrm{kappa2D}(\mathrm{cycloid}(a, b, t), t), assume = real)$

$$\kappa = \frac{\left|-b^2 + b\cos(t)\,a\right|}{\left(a^2 - 2\,b\cos(t)\,a + b^2\right)^{(3/2)}}$$

scalarplot(kappa2D(cycloid(1, 2, t), t), $t = 0 .. 2\,\pi$, *title* = "Curvature of Cycloid(1,2,t)", *labels* = ["t", "k"])



curvatureplot3D(cycloid(1, 2, *t*), kappa2D(cycloid(1, 2, *t*), *t*), *t* = –π .. 3 π, "Curvature of Cycloid(1,2,t)", *axes* = *normal*)



Deltoid

$deltoid = \mathrm{mat}(\mathrm{deltoid}(a, t))$

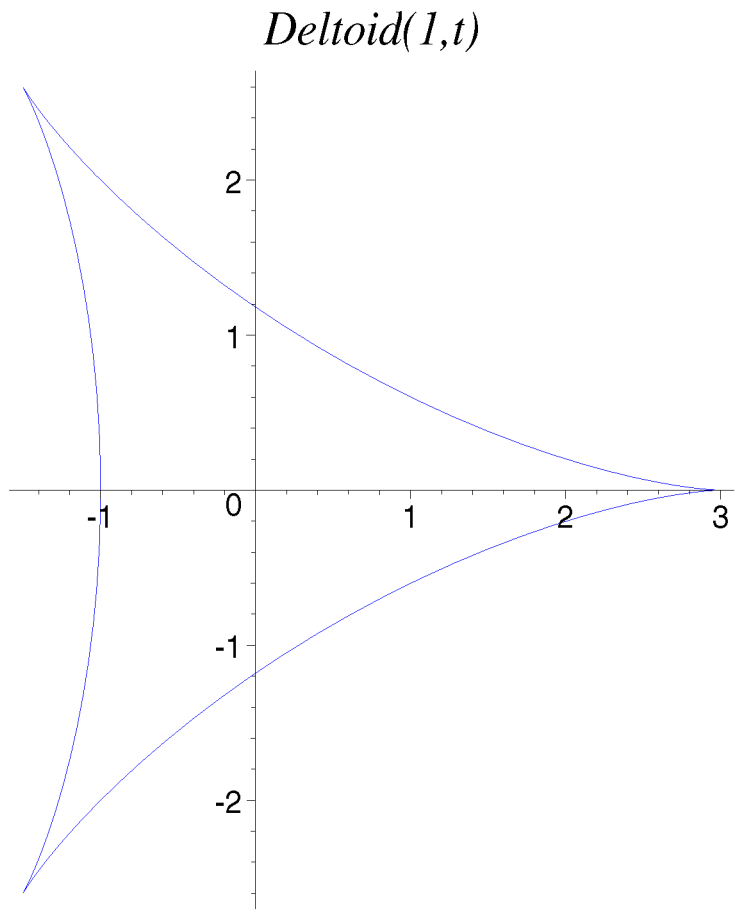
$$deltoid = \begin{bmatrix} 2\,a\cos(t)\,(\cos(t)+1) - a \\ 2\,a\sin(t)\,(1-\cos(t)) \end{bmatrix}$$

$\mathrm{ParametrizedToImplicit}(\mathrm{deltoid}(a, t), ['x', 'y'], t)$

$\mathrm{map}(collect, \mathrm{lhs}(\%), a, factor) = \mathrm{rhs}(\%)$

$$\left(-27\,a^4 + 18\,(x-y)\,(x+y)\,a^2 - 8\,x^3\,a + (x^2+y^2)^2\right)$$
$$\left(-27\,a^4 + (18\,x^2 + 18\,y^2)\,a^2 - 8\,x\,(-3\,y^2 + x^2)\,a + (x^2+y^2)^2\right) = 0$$

$\mathrm{curveplot2D}(\mathrm{deltoid}(1, t), t = 0 .. 2\,\pi, axes = normal, scaling = constrained, title = \text{"Deltoid(1,t)"})$

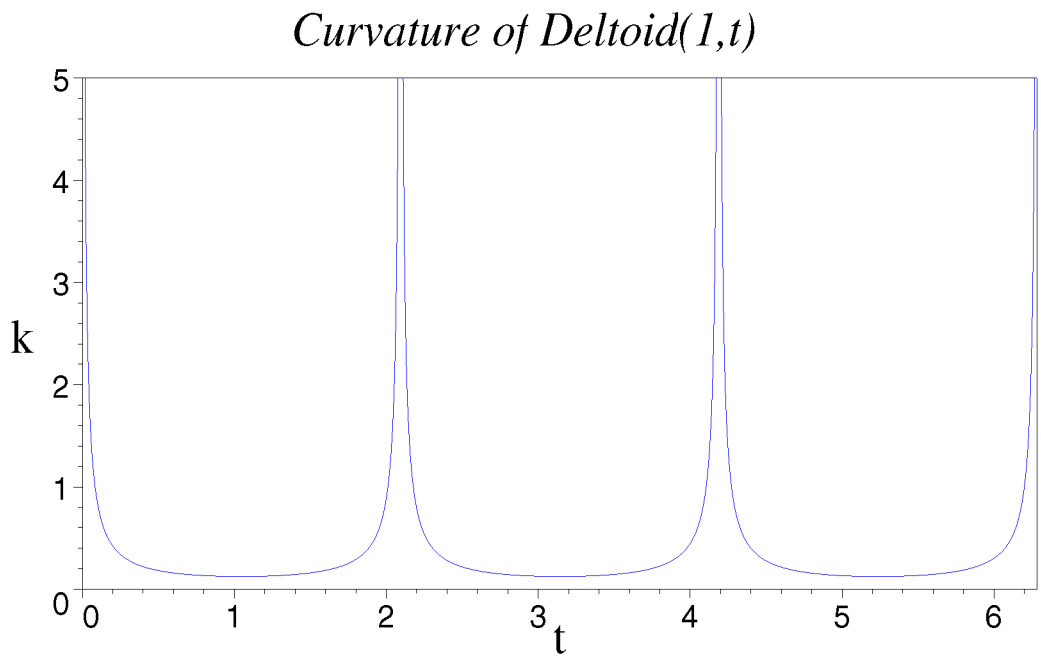


Curvature

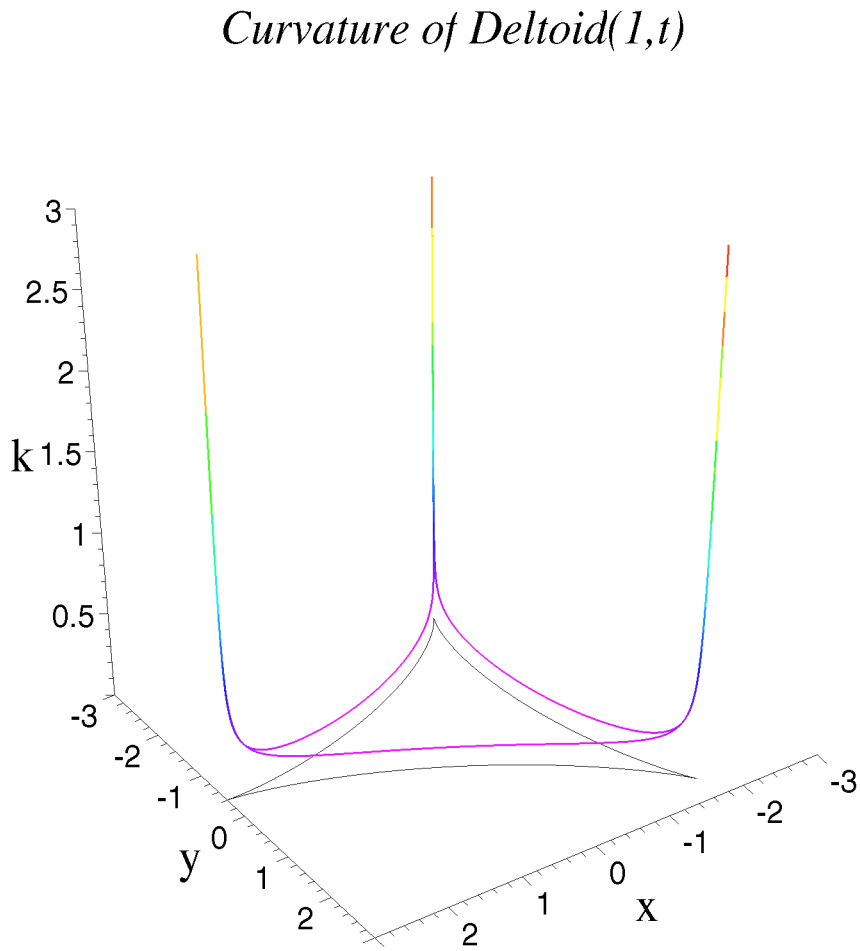
$\kappa = \mathrm{simplify}(\mathrm{kappa2D}(\mathrm{deltoid}(a, t), t), assume = real)$

$$\kappa = \frac{1}{8}\,\frac{\sqrt{2}\,\mathrm{signum}(a)\,\mathrm{signum}(1+2\cos(t))}{a\,(1+2\cos(t))\,\sqrt{1-\cos(t)}}$$

scalarplot(kappa2D(deltoid(1, *t*), *t*), *t* = 0 .. 2 π, *view* = 0 .. 5, *numpoints* = 1000, *labels* = ["t", "k"], *title* = "Curvature of Deltoid(1,t)")



curvatureplot3D(deltoid(1, *t*), kappa2D(deltoid(1, *t*), *t*), *t* = 0 .. 2 π, "Curvature of Deltoid(1,t)", *view* = [−3 .. 3, −3 .. 3, 0 .. 3], *numpoints* = 500, *axes* = *framed*)




```
save "d:/dynamics/Curves2D.m"
read "d:/dynamics/Curves2D.m"
```